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ACM 95b/100b
Dale Pullin

The Gamma function

The Gamma function $\Gamma[z]$ is defined by Euler's formula

$$\Gamma[z] = \int_0^\infty e^{-t}\, t^{z-1}\, dt, \quad , Re(z) > 0$$

In this formula z is a complex variable and t^{z-1} is interpreted as the principal value. The integral converges only when $Re(z) > 0$.

1. Clearly

$$\Gamma[1] = \int_0^\infty e^{-t}\, dt = 1$$

2. When $z = n+1$, $n = 0, 1, 2, 3....$, then

$$\Gamma[n+1] = \int_0^\infty e^{-t}\, t^n\, dt,$$

   Repeated integration by parts then shows that

$$\Gamma[n+1] = n!, \quad n = 1, 2, 3, 4.....$$

   Hence $\Gamma[z]$ can be thought of as a generalized factorial function.

3. Integration by parts once in the general formula shows that

$$\Gamma[z+1] = z\,\Gamma[z]$$

   This is called the recurrence formula.

4. Consider $z = x + i\,y$ to be a complex variable. The integral in the Euler formula converges for any y provided $x > 0$. Hence the integral formula for the Gamma function is defined only for the right half plane $Re[z] > 0$

5. The recurrence formula can be used to analytically continue $\Gamma[z]$ to the left half plane $Re[z] < 0$. Consider how to do this. The recurrence relation can be written as

$$\Gamma[z] = \frac{\Gamma[z+1]}{z}$$

   Consider the strip in the complex plane $-1 < Re[z] \le 0$. The integral diverges for z in this region. But $\Gamma[z+1]$ on the RHS of the above equation can be calculated. Hence for each z, except $z = 0$, $\Gamma[z]$ can be calculated in $-1 < Re[z] \le 0$.

When $z = 0$, the formula fails to give a finite value of $\Gamma[0]$. Hence $z = 0$ is a singularity of the complex function $\Gamma[z]$.

6. This method can be repeated for the strip $-2 < Re[z] \le -1$, and then ... for $-(n+1) < Re[z] \le -n$, except at the points $z = -n$, where the recurrence formula will fail to give a finite result. The points

$$z = 0, -1, -2, -3.....$$

are singularities of $\Gamma[z]$ in the complex z-plane.

7. What is the type of these singularities? To see this, write

$$\mathrm{e}^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n$$

In the integral definition of $\Gamma[z]$, this gives

$$\begin{aligned}\Gamma[z] &= \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt \\ &= \lim_{n\to\infty} n^z \int_0^1 (1-\tau)^n\, \tau^{z-1}\, d\tau\end{aligned}$$

For integral n, integration by parts now gives

$$\begin{aligned}\int_0^1 (1-\tau)^n\, \tau^{z-1}\, d\tau &= \left[\frac{1}{z}\,\tau^z\,(1-\tau)^n\right]_0^1 + \frac{n}{z}\int_0^1 (1-\tau)^{n-1}\,\tau^z\, d\tau \\ &= \ldots \\ &= \frac{n\,(n+1).....(1)}{z\,(z+1)......(z+n)}\end{aligned}$$

Hence

$$\begin{aligned}\Gamma[z] &= \lim_{n\to\infty} \frac{n!\, n^z}{z\,(z+1)......(z+n)} \\ &= \frac{1}{z}\prod_1^\infty \left[\left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1}\right]\end{aligned}$$

where $\prod_{n=1}^\infty u_n$ denotes the infinite product $u_1\, u_2\, u_3\, \ldots$. The infinite product in the last equation was used by Gauss as the definition of $\Gamma[z]$. It is convergent and is an analytic function for all z different from $z = 0$ or $z = -n$, where n is a positive integer. From this definition it is clear that $\Gamma[z]$ has simple pole singularities at $z = 0, -1, -2, -3, -4\ldots$.

8. From the above definition, we can obtain

$$\begin{aligned}\frac{1}{\Gamma[z]} &= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\cdots\cdots\left(1+\frac{z}{n}\right)\mathrm{e}^{-z\,\ln n}\right] \\ &= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)\mathrm{e}^{-z}\left(1+\frac{z}{2}\right)\mathrm{e}^{-\frac{1}{2}z}\cdots\cdots\cdots\left(1+\frac{z}{n}\right)\mathrm{e}^{-\frac{1}{n}z}\mathrm{e}^{[1+\frac{1}{2}+\ldots.\frac{1}{n}-\ln n]z}\right] \\ &= z\,\mathrm{e}^{\gamma\, z}\prod_1^\infty\left[\left(1+\frac{z}{n}\right)\mathrm{e}^{-\frac{1}{n}z}\right]\end{aligned}$$

where

$$\gamma = \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right)-\ln n\right]$$
$$= 0.5772156649.....$$

is call the Euler-Mascheroni constant. This infinite product is uniformly convergent in any bounded region of the z-plane. Hence $1/\Gamma[z]$ is an entire function and it has a first-order zero only where $\Gamma[z]$ has a pole. This definition was used by Weierstrass as the definition of $\Gamma[z]$.

9. From the above results, the following useful properties of $\Gamma[z]$ can be established
(i)

$$\Gamma[3/2] = \frac{1}{2}\,\pi^{1/2}$$

(ii)

$$\Gamma[z]\,\Gamma[1-z] = \frac{\pi}{\sin\pi\,z}$$

(ii)

$$\Gamma[z]\,\Gamma[-z] = -\frac{\pi}{z\,\sin\pi\,z}$$

References

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[2] Abramowitz, M. & Stegun, I. A., *Handbook of Mathematical Functions*, Chapter 6. (Dover)

[3] Watson, G.N. *Bessel Functions*