7 Flächenberechnungen mit dem Integral

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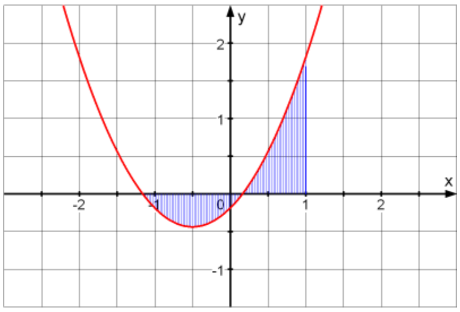
1



Nullstellen:

$$f(x) = x^2 + x - \frac{5}{16} = 0 \quad \Leftrightarrow \quad x = \frac{-1 \pm \sqrt{1 + 4 \cdot 1 \cdot \frac{5}{16}}}{2} = \frac{-1 \pm \frac{3}{2}}{2} \quad \Leftrightarrow \quad x = -\frac{5}{4} \vee x = \frac{1}{2}$$

Erstes Flächenstück:

$$\int_{-\frac{5}{4}}^{\frac{1}{4}} (x^2 + x - \frac{5}{16})dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{5}{16}x\right]_{-\frac{5}{4}}^{\frac{1}{4}} =$$

$$= \left(\frac{1}{3} \cdot \frac{1}{64} + \frac{1}{2} \cdot \frac{1}{16} - \frac{5}{16} \cdot \frac{1}{4}\right) - \left(\frac{1}{3} \cdot (-\frac{125}{64}) + \frac{1}{2} \cdot \frac{25}{16} - \frac{5}{16} \cdot (-\frac{5}{4})\right) = -\frac{1}{24} - \frac{25}{24} = -\frac{13}{12}$$

Also $A_1 = \frac{13}{12}$

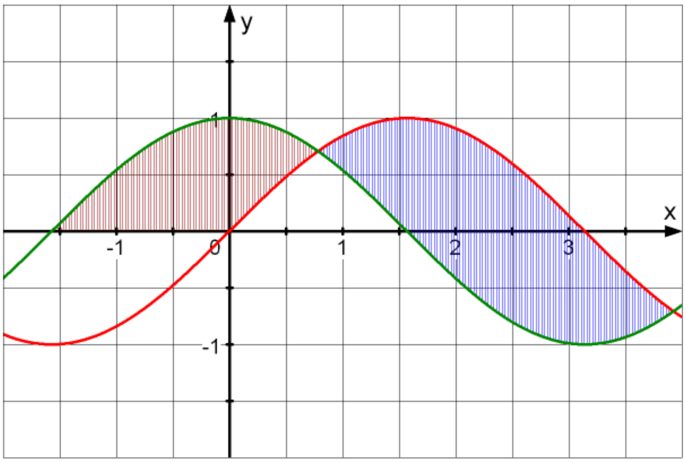
Zweites Flächenstück:

$$\int_{\frac{1}{4}}^{1} (x^2 + x - \frac{5}{16})dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{5}{16}x\right]_{\frac{1}{4}}^{1} = \left(\frac{1}{3} + \frac{1}{2} - \frac{5}{16}\right) - \left(-\frac{1}{24}\right) = \frac{9}{16}$$

Also $A_2 = \frac{9}{16}$

Damit ist $A = \frac{13}{12} + \frac{9}{16} = \frac{79}{48}$

2



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos x dx = \left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{1}{2}\sqrt{2} - (-1) = 1 + \frac{1}{2}\sqrt{2}$$

$$\int_{0}^{\frac{\pi}{4}} \sin x dx = \left[-\cos x\right]_{0}^{\frac{\pi}{4}} = -\frac{1}{2}\sqrt{2} + 1$$

und damit $A_1 = 1 + \frac{1}{2}\sqrt{2} - \left(1 - \frac{1}{2}\sqrt{2}\right) = \sqrt{2}$

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x) dx = \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\right) - \left(-\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\right) = 2\sqrt{2}$$

Also $A_2 = 2\sqrt{2}$

Hat im auf dem Intervall $\left[a; b\right]$ definierte und stetige Funktion f zwischen a und b keine Nullstelle, dann gilt für den Inhalt der Fläche $A_a(b)$, die der Graph von f von a bis b mit der x-Achse einschließt

$$\boxed{A_a(b) = \left|\int_a^b f(x) dx\right|}$$

Liegen zwischen a und b Nullstellen von f, dann müssen die Inhalte die Teilflächen einzeln bestimmt und addiert werden.

3 Flächenberechnungen

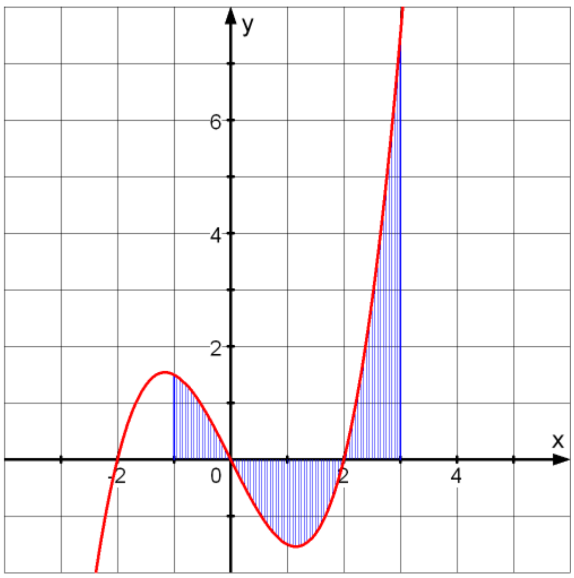
a) $f(x) = \frac{1}{2}x^3 - 2x = 0 \quad \Leftrightarrow \quad \frac{1}{2}x \cdot (x^2 - 4) = 0 \quad \Leftrightarrow \quad x = 0 \vee x = -2 \vee x = 2$

$$\int_{-1}^{0} (\frac{1}{2}x^3 - 2x)dx = \left[\frac{1}{8}x^4 - x^2\right]_{-1}^{0} = -\left(\frac{1}{8} - 1\right) = \frac{7}{8}$$

$$\int_{0}^{2} (\frac{1}{2}x^3 - 2x)dx = \left[\frac{1}{8}x^4 - x^2\right]_{0}^{2} = 2 - 4 = -2$$

$$\int_{2}^{3} (\frac{1}{2}x^3 - 2x)dx = \left[\frac{1}{8}x^4 - x^2\right]_{2}^{3} = \left(\frac{81}{8} - 9\right) - \left(2 - 4\right) = 3\frac{1}{8}$$

$$A_{-1}(3) = \frac{7}{8} + 2 + 3\frac{1}{8} = 6$$



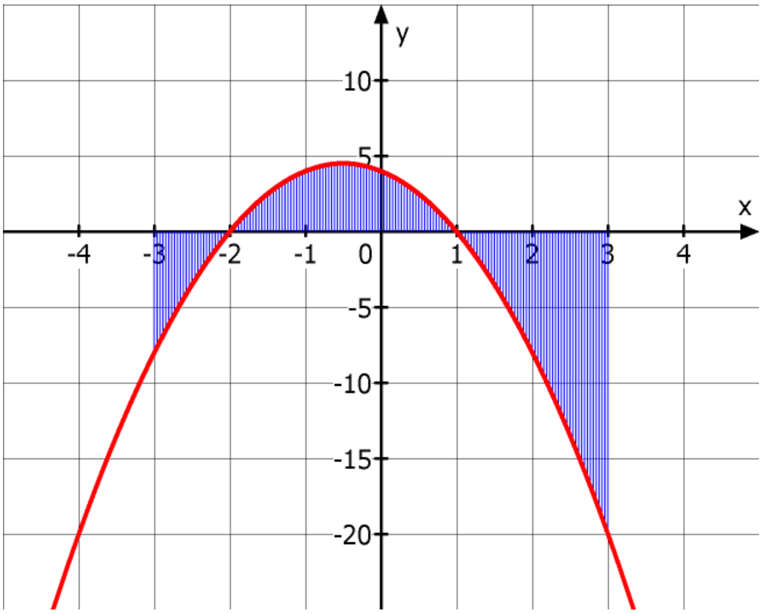
b) $f(x) = -2x^2 - 2x + 4 = 0 \quad \Leftrightarrow \quad x^2 - x - 2 = 0 \quad \Leftrightarrow \quad x = -1 \vee x = 2$

$$\int_{-3}^{-2} (-2x^2 - 2x + 4)dx = \left[-\frac{2}{3}x^3 - x^2 + 4x\right]_{-3}^{-2} = \left(\frac{16}{3} - 4 - 8\right) - \left(18 - 9 - 12\right) = -\frac{11}{3}$$

$$\int_{-2}^{2} (-2x^2 - 2x + 4)dx = \left[-\frac{2}{3}x^3 - x^2 + 4x\right]_{-2}^{2} = \left(-\frac{16}{3} - 4 + 8\right) - \left(\frac{16}{3} - 4 - 8\right) = \frac{16}{3}$$

$$\int_2^3 (-2x^2-2x+4)dx = \left[-\frac{2}{3}x^3-x^2+4x\right]_2^3 = \left(-18-9+12\right) - \left(-\frac{16}{3}-4+8\right) = -\frac{41}{3}$$

$$A_{-3}(3) = \frac{68}{3} = 22\frac{2}{3}$$



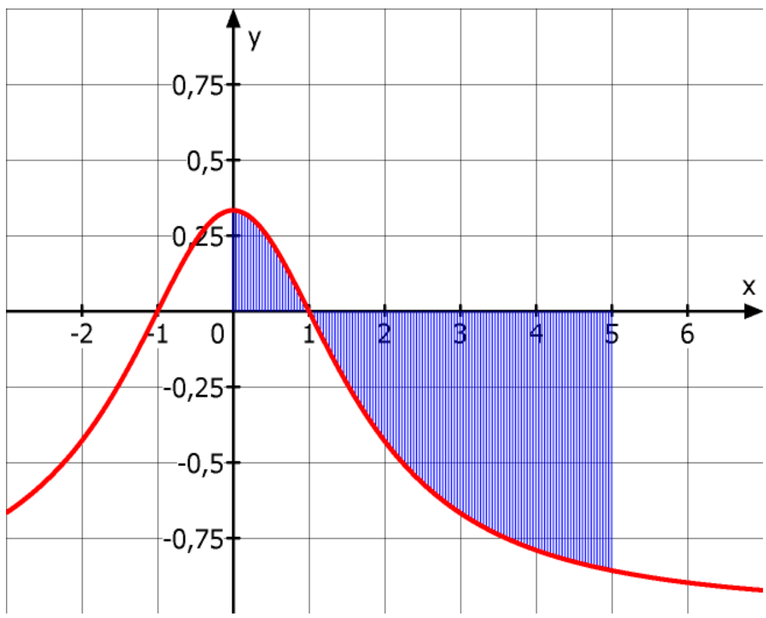
$$\text{c) } f(x) = \frac{4x}{x^2+3} - 1 = 0 \quad \Leftrightarrow \quad x^2-4x+3 = 0 \quad \Leftrightarrow \quad x = 1 \vee x = 3$$

$$\int_0^1 \left(\frac{4x}{x^2+3}-1\right)dx = \left[2\cdot\ln(x^2+3)-x\right]_0^1 = \left(2\cdot\ln 4-1\right) - \left(2\cdot\ln 3-0\right) = 2\cdot\ln\frac{4}{3}-1$$

$$\int_1^3 \left(\frac{4x}{x^2+3}-1\right)dx = \left[2\cdot\ln(x^2+3)-x\right]_1^3 = \left(2\cdot\ln 12-3\right) - \left(2\cdot\ln 4-1\right) = 2\cdot\ln 3-2$$

$$\int_3^5 \left(\frac{4x}{x^2+3}-1\right)dx = \left[2\cdot\ln(x^2+3)-x\right]_3^5 = \left(2\cdot\ln 28-5\right) - \left(2\cdot\ln 12-3\right) = 2\cdot\ln\frac{7}{3}-2$$

$$A_0(5) = -\left(2\cdot\ln\frac{4}{3}-1\right) + \left(2\cdot\ln 3-2\right) - \left(2\cdot\ln\frac{7}{3}-2\right) = 1+\ln\frac{27}{28} \approx 0{,}9636$$



d) $f(x) = \frac{1}{2}x^3 - 1{,}5x^2 - 2x = 0 \quad \Leftrightarrow \quad x^3 - 3x^2 - 4x = 0 \quad \Leftrightarrow \quad x \cdot (x^2 - 3x - 4) = 0$

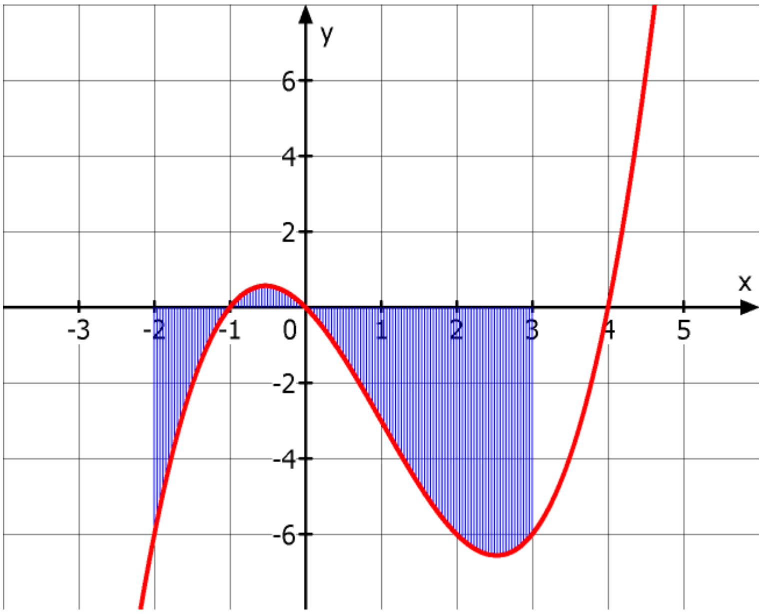
$x = -1 \ \vee \ x = 0 \ \vee \ x = 4$

$$\int_{-2}^{-1} \left(\frac{1}{2}x^3 - 1{,}5x^2 - 2x\right)dx = \left[\frac{1}{8}x^4 - \frac{1}{2}x^3 - x^2\right]_{-2}^{-1} = \left(\frac{1}{8} + \frac{1}{2} - 1\right) - \left(2 + 4 - 4\right) = -2\frac{3}{8}$$

$$\int_{-1}^{0} \left(\frac{1}{2}x^3 - 1{,}5x^2 - 2x\right)dx = \left[\frac{1}{8}x^4 - \frac{1}{2}x^3 - x^2\right]_{-1}^{0} = 0 - \left(\frac{1}{8} + \frac{1}{2} - 1\right) = \frac{3}{8}$$

$$\int_{0}^{3} \left(\frac{1}{2}x^3 - 1{,}5x^2 - 2x\right)dx = \left[\frac{1}{8}x^4 - \frac{1}{2}x^3 - x^2\right]_{0}^{3} = \frac{81}{8} - \frac{27}{2} - 9 = -12\frac{3}{8}$$

$A_{-2}(3) = 15\frac{1}{8}$



e) $f(x) = 2\cdot\sin(2x) - 2 \leq 0$

$$\int_{-1}^{2} (2\cdot\sin(2x) - 2)dx = \Big[-\cos(2x) - 2x\Big]_{-1}^{2} = \Big(-4 - \cos 4\Big) - \Big(2 - \cos(-2)\Big) =$$

$$= \cos 2 - \cos 4 - 6 \approx -5{,}7625$$

f) $$\int_{-3}^{0} 0{,}25\cdot\sqrt{x+4}\,dx = \left[\frac{1}{6}\sqrt{(x+4)^3}\right]_{-3}^{0} = \frac{4}{3} - \frac{1}{6} = \frac{7}{6}$$

g) $f(x) = 0 \quad \Leftrightarrow \quad x = 0{,}5$

$$\int_{-1}^{0,5} -2\cdot(x-0{,}5)\cdot e^{x-x^2}dx = \left[e^{x-x^2}\right]_{-1}^{0,5} = e^{0,25} - e^{-2}$$

$$\int_{0,5}^{1} -2\cdot(x-0{,}5)\cdot e^{x-x^2}dx = \left[e^{x-x^2}\right]_{0,5}^{1} = e^{0} - e^{0,25}$$

$$A_{-1}(1) = \left(e^{0,25} - e^{-2}\right) - \left(1 - e^{0,25}\right) = 2\cdot e^{0,25} - e^{-2} - 1 \approx 1{,}43272$$

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4 Flächeneinschluss mit der x-Achse

a) $\int_0^2 [x\cdot(4-x^2)]dx = \int_0^2 (4x-x^2)dx = \left[2x^2-\frac{1}{4}x^4\right]_0^2 = 4 \quad \Rightarrow \quad A = 2\cdot 4 = 8$

b) $\int_1^3 [(x-2)^2-1]dx = \int_1^3 (x^2-4x+3)dx = \left[\frac{1}{3}x^3-2x^2+3x\right]_1^3 = \left(9-18+9\right) - \left(\frac{1}{3}-2+3\right) =$

$= -\frac{4}{3} \quad \Rightarrow \quad A = \frac{4}{3}$

c) $\int_0^{\sqrt{2}} (-x^2+2)dx = \left[-\frac{1}{3}x^3+2x\right]_0^{\sqrt{2}} = -\frac{2}{3}\sqrt{2}+2\sqrt{2} = \frac{4}{3}\sqrt{2} \quad \Rightarrow \quad A = \frac{8}{3}\sqrt{2}$

d) $\int_{2-\sqrt{3}}^{2+\sqrt{3}} (x+\frac{1}{x}-4)dx = \left[\frac{1}{2}x^2+\ln x-4x\right]_{2-\sqrt{3}}^{2+\sqrt{3}} \underset{\text{endlich lange Rechnung}}{=} \ln\left(4\sqrt{3}+7\right)-4\sqrt{3} \approx -4{,}3$

$A = 4\sqrt{3}-\ln\left(4\sqrt{3}+7\right)$

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e) $\int_0^2 (x^4-4x^2)dx = \left[\frac{1}{5}x^5-\frac{4}{3}x^3\right]_0^2 = -4\frac{4}{15} \quad \Rightarrow \quad A = 8\frac{8}{15}$

f) $\int_{0,25}^{2,25} (\frac{1,5}{2\sqrt{x}}+\sqrt{x}-2)dx = \left[1,5x^{0,5}+\frac{2}{3}x^{1,5}-2x\right]_{0,25}^{2,25} = -\frac{1}{3} \quad \Rightarrow \quad A = \frac{1}{3}$

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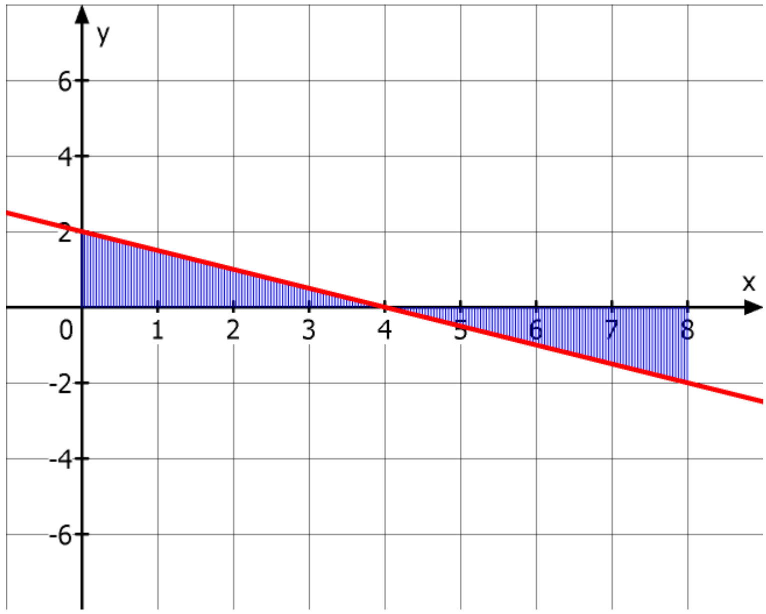
5 Flächengleichheit

a) $\int_0^k (-0,5x+2)dx = \left[-\frac{1}{4}x^2+2x\right]_0^k = -\frac{1}{4}k^2+2k = 0 \quad \Rightarrow \quad <x = 0> \ \vee \ x = 8$



b) $\int_0^k (x^2-2x-3)dx = \left[\frac{1}{3}x^3-x^2-3x\right]_0^k = \frac{1}{3}k^3-k^2-3k = 0$

$k\cdot(\frac{1}{3}k^2-k-3) = 0 \quad \Leftrightarrow \quad k = 0 \vee k^2-3k-9 = 0 \quad \Leftrightarrow \quad k = 0 \vee k = \frac{3 \pm 3\sqrt{5}}{2}$

Sinnvolle Lösung: $k = \frac{3}{2}+\frac{3}{2}\sqrt{5}$

c) $\int_0^k (x^3-1)dx = \left[\frac{1}{4}x^4-x\right]_0^k = \frac{1}{4}k^4-k = 0$

Sinnvolle Lösung: $k = \sqrt[3]{4}$

d) $\int_0^k (x^3+\frac{3}{2}x-2)dx = \left[\frac{1}{4}x^4+\frac{3}{4}x^2-2x\right]_0^k = \frac{1}{4}k^4+\frac{3}{4}k^2-2k = 0$

$\Rightarrow k = 0 \vee k^3+3k-8 = 0 \quad k \approx 1{,}51$

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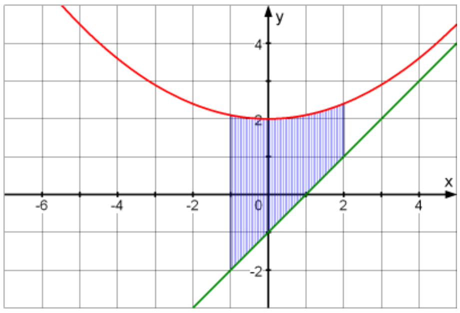
6 Fläche zwischen zwei Graphen

a)



$$\int_{-1}^{2}(0{,}1x^2+2-x+1)dx = \left[\frac{1}{30}x^3-\frac{1}{2}x^2+3x\right]_{-1}^{2} = \left(\frac{4}{15}-2+6\right)-\left(-\frac{1}{30}-\frac{1}{2}-3\right) = 7{,}8$$

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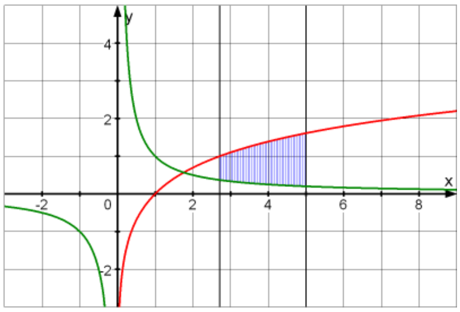
b)



$$\int_{e}^{5}(\ln x-\frac{1}{x})dx = \left[x\cdot\ln x-x-\ln x\right]_{e}^{5} = \left(5\cdot\ln 5-5-\ln 5\right)-\left(e-e-1\right) = 4\cdot\ln 5-4$$

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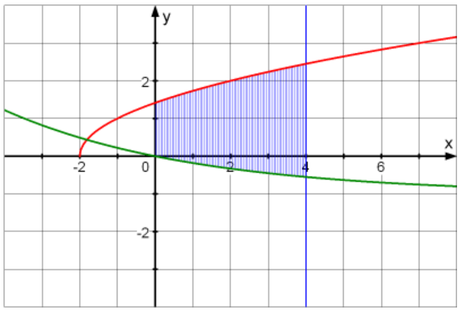
c)



$$\int_{0}^{4}(\sqrt{x+2}\ -\ e^{-0{,}2x}+1)dx = \left[\frac{2}{3}\cdot\sqrt{(x+2)^3}\ +5\cdot e^{-0{,}2x}+x\right]_{0}^{4} =$$

$$= \left(4\sqrt{6}+5e^{-0{,}8}+4\right)-\left(\frac{4}{3}\sqrt{2}\ +5\right)\approx 9{,}16$$

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7 Eingeschlossene Fläche

a) $g'(x) = x \quad\Rightarrow\quad g'(3) = 3$

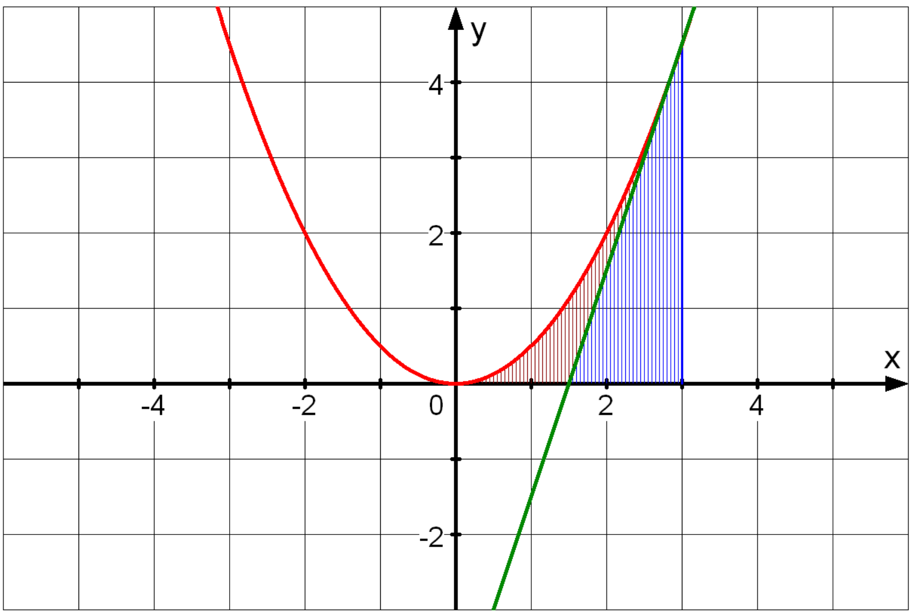
Tangentengleichung: $y = 3 \cdot (x-3) + 4{,}5 = 3x - 4{,}5$

Schnittstelle der Tangente mit der x-Achse: $x = 1{,}5$

$$A_1 = \int_0^3 0{,}5x^2 dx = \left[\frac{1}{6}x^3\right]_0^3 = 4{,}5$$

$$A_2 = \frac{1}{2} \cdot 1{,}5 \cdot 4{,}5 = \frac{27}{8} = 3{,}375$$

$$A = A_1 - A_2 = 1{,}125$$



b) $g'(x) = 4 \cdot (x-2)^3 \quad \Rightarrow \quad g'(0) = -32$

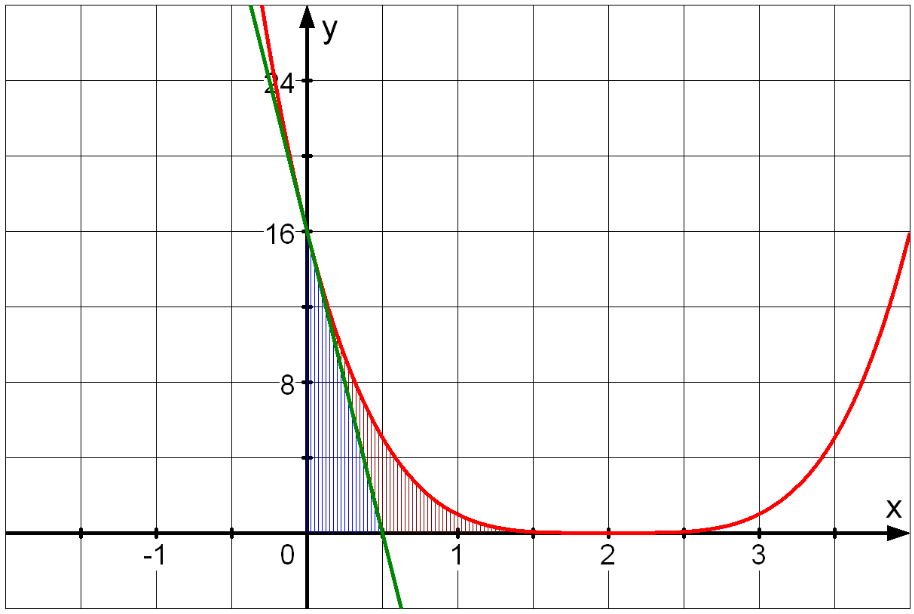
Tangentengleichung: $y = -32 \cdot x + 16$

Schnittstelle der Tangente mit der x-Achse: $x = 0{,}5$

$$A_1 = \int_0 (x-2)^4 dx = \left[\frac{1}{5} \cdot (x-2)^5\right]_0^2 = \frac{32}{5} = 6{,}4$$

$$A_2 = \frac{1}{2} \cdot 0{,}5 \cdot 16 = 4$$

$$A = A_1 - A_2 = 2{,}4$$



c) $g'(x) = -2 \cdot x^{-3} \quad \Rightarrow \quad g'(0{,}5) = -16$

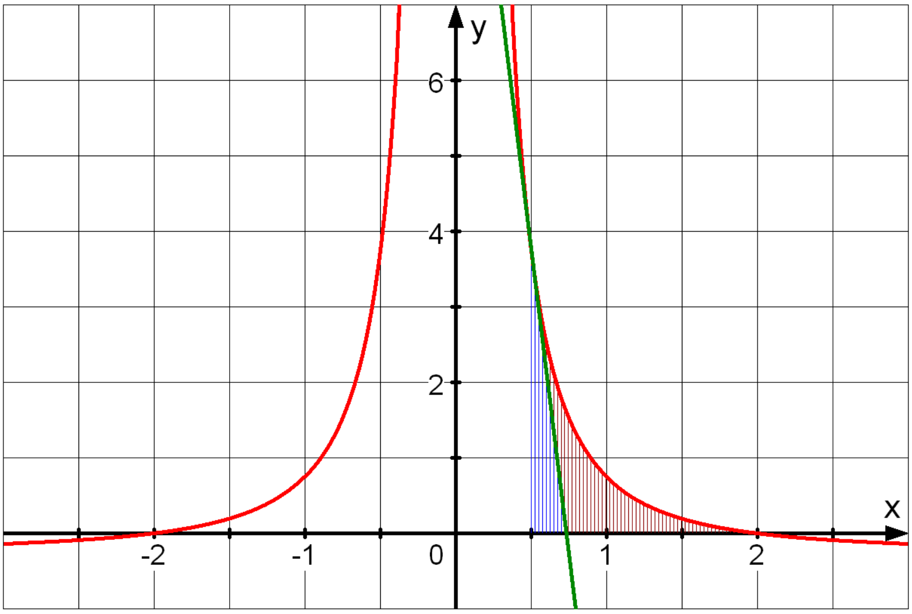
Tangentengleichung: $y = -16 \cdot (x - 0{,}5) + 3{,}75 = -16x + 11{,}75$

Schnittstelle der Tangente mit der x-Achse: $x = \frac{47}{64}$

$$A_1 = \int_{0,5}^{2} (x^{-2} - 0{,}25)dx = \left[-x^{-1} - 0{,}25x \right]_{0,5}^{2} = \left(-\frac{1}{2} - 0{,}5 \right) - \left(-2 - 0{,}125 \right) = 1{,}125$$

$$A_2 = \frac{1}{2} \cdot \frac{15}{64} \cdot \frac{15}{4} = \frac{225}{512}$$

$$A = A_1 - A_2 = \frac{351}{512}$$



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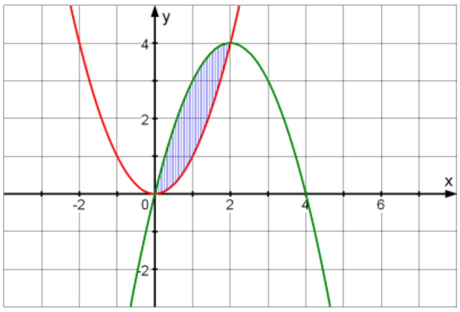
8 Fläche zwischen zwei Graphen

a)

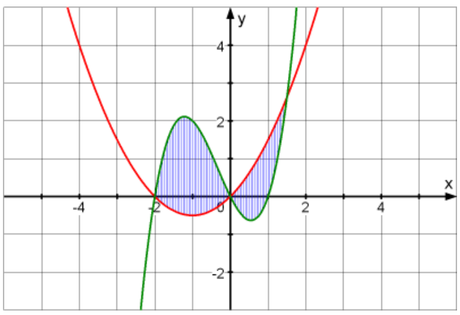


Schnittstellen:

$$x^2 = -x^2+4x \quad \Leftrightarrow \quad 2x^2-4x = 0 \quad \Leftrightarrow \quad 2x\cdot(x-2) = 0 \quad \Leftrightarrow \quad x = 0 \vee x = 2$$

$$\int_0^2 [x^2-(-x^2+4x)]dx = \int_0^2 (2x^2-4x)dx = \left[\frac{2}{3}x^3-2x^2\right]_0^2 = -\frac{8}{3} \quad \Rightarrow \quad A = \frac{8}{3}$$

b)



Schnittstellen:

$$x + 0{,}5x^2 = x^3 + x^2 - 2x \quad \Leftrightarrow \quad x^3 + 0{,}5x^2 - 3x = 0 \quad \Leftrightarrow \quad x\cdot(x^2 + 0{,}5x - 3) = 0$$

$$\Leftrightarrow \quad x = 0 \ \vee \ x^2 + 0{,}5x - 3x = 0 \quad \Leftrightarrow \quad x = 0 \ \vee \ x = -2 \ \vee \ x = 1{,}5$$

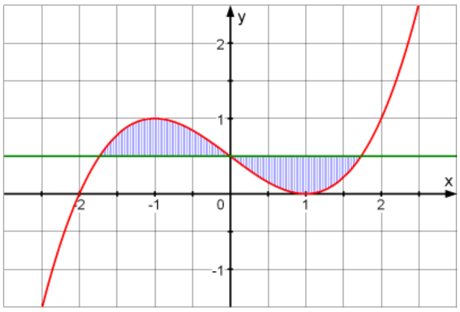
$$\int_{-2}^{0} [(x^3 + x^2 - 2x) - (x + 0{,}5x^2)]dx = \int_{-2}^{0} (x^3 + 0{,}5x^2 - 3x)dx = \left[\frac{1}{4}x^4 + \frac{1}{6}x^3 - \frac{3}{2}x^2\right]_{-2}^{0} =$$

$$= 0 - \left(4 - \frac{4}{3} - 6\right) = \frac{10}{3}$$

$$\int_{0}^{1{,}5} (x^3 + 0{,}5x^2 - 3x)dx = \left[\frac{1}{4}x^4 + \frac{1}{6}x^3 - \frac{3}{2}x^2\right]_{0}^{1{,}5} = \frac{81}{64} + \frac{9}{16} - \frac{27}{8} = -\frac{99}{64}$$

$$A = \frac{10}{3} + \frac{99}{64} = \frac{937}{192} = 4\frac{169}{192}$$

c)



Schnittstellen:

$$\frac{1}{4}\cdot(x+2)\cdot(x-1)^2 = 0{,}5 \quad \Leftrightarrow \quad (x+2)\cdot(x^2 - 2x + 1) = 2 \quad \Leftrightarrow \quad x^3 - 3x = 0$$

$$\Leftrightarrow \quad x = -\sqrt{3} \ \vee \ x = 0 \ \vee \ x = \sqrt{3}$$

$$\int_{0}^{\sqrt{3}} [f(x) - g(x)]dx = \frac{1}{4}\cdot\int_{0}^{\sqrt{3}} (x^3 - 3x)dx = \frac{1}{4}\cdot\left[\frac{1}{4}x^4 - \frac{3}{2}x^2\right]_{0}^{\sqrt{3}} = \frac{9}{16} - \frac{9}{8} = -\frac{9}{16}$$

$$A = 2\cdot\frac{9}{16} = 1{,}125$$

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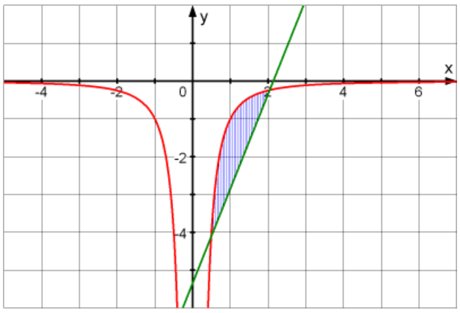
d)



Schnittstellen:

$$-\frac{1}{x^2} = 2{,}5x - 5{,}25 \quad \Rightarrow \quad 2{,}5x^3 - 5{,}25x + 1 = 0$$

Erraten: $x = 2 \quad \rightarrow \quad (2{,}5x^3 - 5{,}25x^2 - 1):(x-2) = 2{,}5x^2 - 0{,}25x - 0{,}5$

$$2{,}5x^2 - 0{,}25x - 0{,}5 = 0 \quad \Leftrightarrow \quad x = -0{,}4 \vee x = 0{,}5$$

$$\int_{0,5}^{2} (2{,}5x - 5{,}25 + \frac{1}{x^2})dx = \left[\frac{5}{4}x^2 - \frac{21}{4}x - \frac{1}{x}\right]_{0,5}^{2} = \left(5 - 10{,}5 - 0{,}5\right) - \left(\frac{5}{16} - \frac{21}{8} - 2\right) =$$

$$= -1{,}6875 \quad \Rightarrow \quad A = 1{,}6875$$

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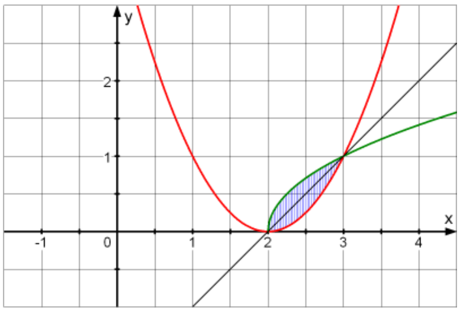
e)



Schnittstellen: $x = 2 \vee x = 3$

$$\int_{0}^{1} (x - x^2)dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{0}^{1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad \Rightarrow \quad A = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

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9 Flächenstücke

a) Gleichungen der beiden Geraden: $y = \frac{1}{2}x - \frac{1}{2}$ und wie angegeben $y = x - 4$.

Durch Einsetzen lässt sich zeigen, dass $\left(-1 \mid 1\right)$, $\left(3 \mid -1\right)$ und $\left(7 \mid 3\right)$ die "Eckpunkte" des Flächenstücks sind.

$$A_1 = \int_{-1}^{3}(\sqrt{x+2}+\frac{1}{2}x-\frac{1}{2})dx = \left[\frac{2}{3}(x+2)^{\frac{3}{2}}+\frac{1}{4}x^2-\frac{1}{2}x\right]_{-1}^{3} =$$

$$= \left(\frac{10}{3}\sqrt{5}+\frac{9}{4}-\frac{3}{2}\right) - \left(\frac{2}{3}+\frac{1}{4}+\frac{1}{2}\right) = \frac{10}{3}\sqrt{5} - \frac{2}{3}$$

$$A_2 = \int_{3}^{7}(\sqrt{x+2}-x+4)dx = \left[\frac{2}{3}(x+2)^{\frac{3}{2}}-\frac{1}{2}x^2+4x\right]_{3}^{7} =$$

$$= \left(18-\frac{49}{2}+28\right)-\left(\frac{10}{3}\sqrt{5}-\frac{9}{2}+12\right) = 14-\frac{10}{3}\sqrt{5}$$

$$A = A_1+A_2 = 13\frac{1}{3}$$

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b) Tangente: $y = \frac{1}{2}\cdot(x-5)+1{,}5 = \frac{1}{2}x-1$

Parabel: $y = \frac{1}{8}\cdot(x-3)^2+1 = \frac{1}{8}x^2-\frac{3}{4}x+\frac{27}{8}$

$$A = \int_{0}^{5}(\frac{1}{8}x^2-\frac{3}{4}x+\frac{17}{8}-\frac{1}{2}x+1)dx = \int_{0}^{5}(\frac{1}{8}x^2-\frac{5}{4}x+\frac{25}{8})dx = \left[\frac{1}{24}x^3-\frac{5}{8}x^2+\frac{25}{8}x\right]_{0}^{5} =$$

$$= 125\cdot\left(\frac{1}{24}-\frac{1}{8}+\frac{1}{8}\right) = \frac{125}{24} = 5\frac{5}{24}$$

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10 Parameterabhängige Fläche

a) $A(t) = \int_{1}^{2}\frac{t}{x^2}dx = \int_{1}^{2}t\cdot x^{-2}dx = \left[-t\cdot x^{-1}\right]_{1}^{2} = -\frac{t}{2}+t = \frac{t}{2}$

$\frac{t}{2} = 8 \quad\Rightarrow\quad t = 16$

b) Nullstellen: $x^2-t^2 = 0 \quad\Leftrightarrow\quad x = -t \vee x = t$

$$\int_0^t (x^2 - t^2)dt = \left[\frac{1}{3}x^3 - t^2x\right]_0^t = \frac{1}{3}t^3 - t^3 = -\frac{2}{3}t^3 \quad \Rightarrow \quad A(t) = \frac{4}{3}t^3$$

$$\frac{4}{3}t^3 = 36 \quad \Rightarrow \quad t = 3$$

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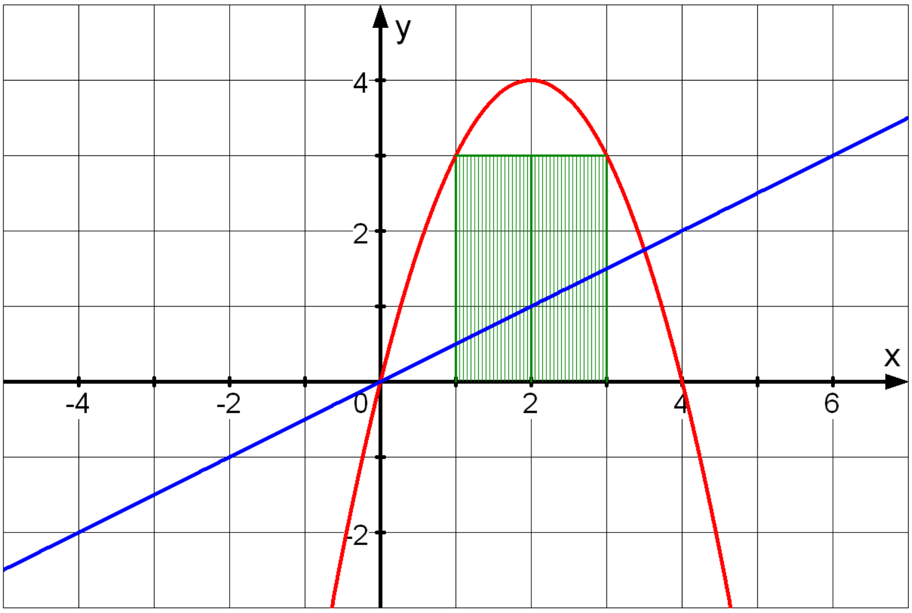
11 Parabel

a) $p(2) = 2\cdot(4-2) = 4$ und damit $S\left(2 \mid 4\right)$



b) $A = \int_0^4 x\cdot(4-x)dx = \int_0^4 (4x - x^2)dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{32}{3} = 10\frac{2}{3}$

$$\frac{A - U_4}{A} = \frac{10\frac{2}{3} - 6}{10\frac{2}{3}} = \frac{7}{16} = 43{,}75\%$$

c) s.o.

d) Die Fläche liegt im 4.Quadranten, wenn Steigung der Geraden nichtnagativ und kleiner als die Steigung der Tangente an den Graphen von p im Punkt $O\left(0 \mid 0\right)$ ist

$p'(x) = 4 - 2x \quad \Rightarrow \quad p'(0) = 4$

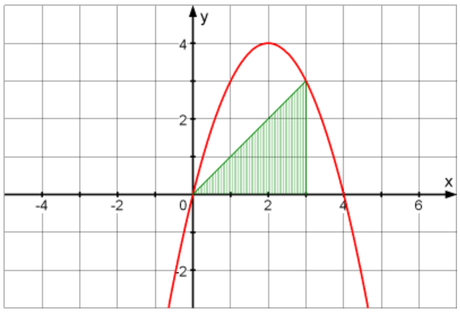
Es muss also $0 \le a < 4$ sin, damit die eingeschlossene Fläche kleiner im I.Quadranten liegt.

e) Abszissen (= x-Koordinaten) der Schnittpunkte:

$$4x - x^2 = ax \quad \Leftrightarrow \quad x \cdot (4 - a - x) = 0 \quad \Leftrightarrow \quad x = 0 \vee x\,) = a - 4$$

$$A(a) = \int_0^{4-a} (4x - x^2 - ax)dx = \int_0^{4-a} [(4-a)\cdot x - x^2]dx = \left[(4-a)\cdot\frac{1}{2}x^2 - \frac{1}{3}\cdot x^3\right]_0^{4-a} =$$

$$(4-a)\cdot\frac{1}{2}\cdot(4-a)^2 - \frac{1}{3}\cdot(4-a)^3 = \frac{1}{6}\cdot(4-a)^3$$



$A_{OPQ} = 4{,}5$

g) $A(x) = \frac{1}{2}\cdot x \cdot p(x) = 2x^2 - \frac{1}{2}x^3 \quad \Rightarrow \quad A'(x) = 4x - \frac{3}{2}x^2 = 0 \quad \Rightarrow \quad x = 0 \vee x = \frac{8}{3}$

$A''(x) = 4 - 3x \quad \Rightarrow \quad A''(\frac{8}{3}) = -4 < 0$

Also liegt für $x = \frac{8}{3}$ ein Maximum des Flächeninhalts vor.

Wegen $A(0) = A(4)$ erhält man für $x = 0$ oder $x = 4$ Minima des Inhalts.

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12 Beton

a) $\int_0^{40} \frac{1}{20}x^2 dx = \left[\frac{1}{60}x^3\right]_0^{40} = \frac{3200}{3}$

Querschnittsfläche: $A = 100\cdot 20 + 2\cdot 10\cdot 80 + 2\cdot\frac{3200}{3} = 5733\frac{1}{3}\left(cm^3\right)$

Ein Betonstein wiegt 1,3 t.

b) $f(x) = a\cdot x^2 + 250 \quad f(100) = a\cdot 100^2 + 250 = 0 \quad \Rightarrow \quad a = -\frac{1}{40}$

$$\int_0^{100} \left(-\frac{1}{40}x^2 + 250\right)dx = \left[-\frac{1}{120}x^3 + 250x\right]_0^{100} = 16666\frac{2}{3}$$

Querschnittfläche: $A = 400 \cdot 350 - 100 \cdot 100 - 2 \cdot 16666\frac{2}{3} = 96666\frac{2}{3} \left(\text{cm}^2\right)$

Man benötigt 97,7 m^3 Beton.

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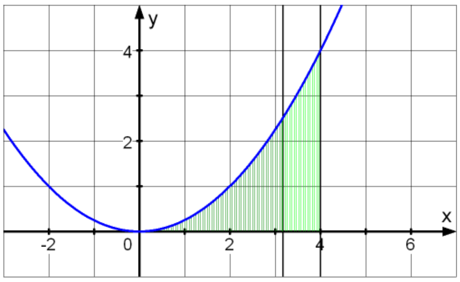
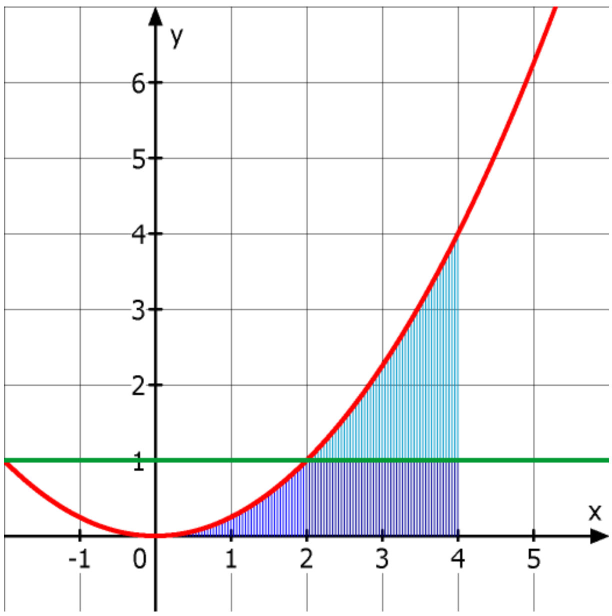
13 Flächengleichheit

a) $\int_0^a \frac{1}{4}x^2dx = \int_a^4 \frac{1}{4}x^2dx \Leftrightarrow \frac{1}{12}a^3 = \frac{1}{12} \cdot 4^3 - \frac{1}{12} \cdot a^3$

$\Leftrightarrow a^3 = 32 \Rightarrow a = 2\sqrt[3]{4}$



b)



$\frac{1}{4}x^2 = a \Rightarrow x = -2\sqrt{a} \vee x = 2\sqrt{a}$

$$\int_{2\sqrt{a}}^{4} (\frac{1}{4}x^2 - a)dx = \left[\frac{1}{12}x^3 - ax\right]_{2\sqrt{a}}^{4} = \left(\frac{16}{3} - 4a\right) - \left(\frac{2}{3}a\sqrt{a} - 2a\sqrt{a}\right) = \frac{8}{3}$$

$$\frac{4}{3}a\sqrt{a} - 4a + \frac{8}{3} = 0 \quad \Leftrightarrow \quad a\sqrt{a} - 3a + 2 = 0 \quad \Rightarrow \quad a = 1$$

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14 Funktionenscharen

a) $f_a(x) = \frac{1}{a}x^2 - a = 0 \quad \Leftrightarrow \quad x^2 = a \quad \Leftrightarrow \quad x = -a \vee x = a$

b) $f_a(-x) = \frac{1}{a}\cdot(-x)^2 - a = \frac{1}{a}\cdot x^2 - a = f_a(x)$

$h_a(-x) = \frac{1}{3}\cdot(-x)^2 - \frac{1}{3}a^2 = \frac{1}{3}\cdot x^2 - \frac{1}{3}a^2 = h_a(x)$

c) $h_a(x) = \frac{1}{3}x^2 - \frac{1}{3}a^2 = 0 \quad \Leftrightarrow \quad x = -a \vee x = a$

d) $\int_0^a [(\frac{1}{a}x^2 - a) - (\frac{1}{3}x^2 - \frac{1}{3}a^2)]dx = \left[\frac{1}{3a}x^3 - ax - \frac{1}{9}x^3 + \frac{1}{3}a^2x\right]_0^a = \frac{a^2}{3} - a^2 - \frac{1}{9}a^3 + \frac{1}{3}a^3 =$

$= \frac{2}{9}a^3 - \frac{2}{3}a^2$

$A(a) = 2\cdot\left|\frac{2}{9}a^3 - \frac{2}{3}a^2\right| = \left|\frac{4}{9}a^3 - \frac{4}{3}a^2\right|$

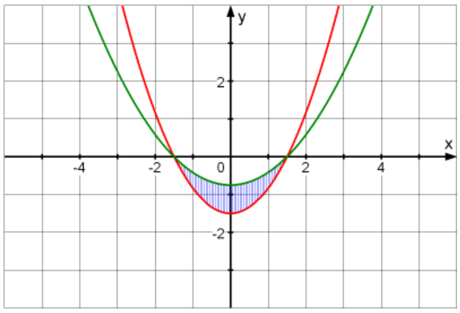
$\frac{4}{9}a^3 - \frac{4}{3}a^2 = 0 \quad \Leftrightarrow \quad a = 0 \vee a = 3$

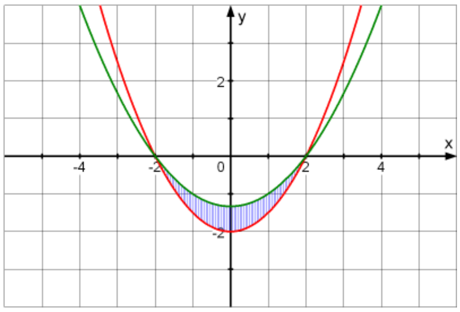
	$-\infty < x < 0$	$0 < x < 3$	$3 < x < \infty$
$\frac{4}{9}a^3 - \frac{4}{3}a^2$	$-$	$-$	$+$

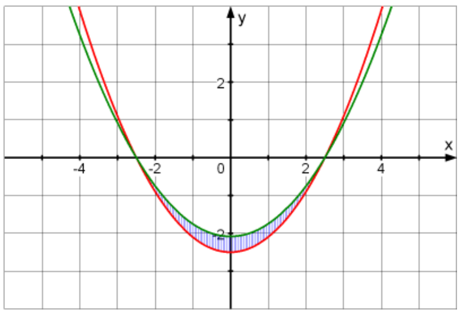
d.h. für $0 < a < 3$ gilt

$A(a) = \frac{4}{3}a^2 - \frac{4}{9}a^3 \quad \Rightarrow \quad A'(a) = \frac{8}{3}a - \frac{4}{3}a^2 = 0 \quad \Rightarrow \quad \langle a = 0\rangle \vee a = a = 2$

Es liegt ein Maximum des Flächinhalts vor.







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15 Unendlich ausgedehnte Flächen

a) $I(a) = \int_1^a \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^a = -\frac{1}{a} + 1 = 1 - \frac{1}{a}$

b) $\lim_{a \to \infty} I(a) = 1$

Die unendlich ausgedehnte Fläche,

die der Graph der Funktion $f : x \to \frac{1}{x^2}$ mit der Geraden $x = 1$ und der x-Achse im

1. Quadranten des Koordinatensystems einschließt hat den Inhalt 1

c) $I(a) = \int_a^4 \sqrt{x}dx = \left[\frac{1}{2}\sqrt{x}\right]_a^4 = 2 - \frac{1}{2}\sqrt{a}$ mit $\lim_{a \to 0+0} I(a) = 1$

Die unendlich ausgedehnte Fläche,

die der Graph der Funktion $f: x \rightarrow \frac{1}{\sqrt{x}}$ mit der Geraden $x = 4$ und den beiden Koordina-

ten-achsen im 1. Quadranten des Koordinatensystems einschließt hat den Inhalt 1

G 16 Steckbriefaufgabe

Ansatz: $f(x) = ax^4 + bx^3 + cx^2 \quad \Rightarrow \quad f'(x) = 4ax^3 + 3bx^2 + 2cx$

(1) $f(2) = 16 \quad \Rightarrow \quad 16a + 8b + 4c = 16 \quad \Leftrightarrow \quad 4a + 2b + c = 4$

(2) $f(3) = 27 \quad \Rightarrow \quad 81a + 27b + 9c = 27 \quad \Leftrightarrow \quad 9a + 3b + c = 3$

(3) $f'(2) = 0 \quad \Rightarrow \quad 32a + 12b + 4c = 0 \quad \Leftrightarrow \quad 8a + 3b + c = 0$

Als Lösung ergibt sich $a = 3$ $b = -16$ und $c = 24$ d.h. $f(x) = 3x^4 - 16x^3 + 24x^2$

Eine Überprüfung der kritischen Punkte ergibt jedoch, dass $H\left(2 \mid 12\right)$ kein Hochpunkt, sondern ein Terrassenpunkt ist!

Es gibt also keine ganzrationale Funktion mit diesen Eigenschaften.