8 Produktregel und Quotientenregel

===

3 Differenzieren mit der Produktregel

a) $f(x) = (x^3 - 2x - 1)\cdot(x^2 + 3) = x^5 - 2x^3 - x^2 + 3x^3 - 6x - 3 = x^5 + x^3 - x^2 - 6x - 3$

$\Rightarrow \quad f'(x) = 5x^4 + 3x^2 - 2x - 6$

Mit der Produktregel:

$f'(x) = (3x^2 - 2)\cdot(x^2 + 3) + (x^3 - 2x - 1)\cdot 2x = 3x^4 - 2x^2 + 9x^2 - 6 + 2x^4 - 4x^2 - 2x =$

$= 5x^4 + 3x^2 - 2x - 6$

----

b) $f(x) = (2x^2 - x + 2)\cdot(0{,}5x^2 - x - 3) = x^4 - 0{,}5x^3 + x^2 - 2x^3 + x^2 - 2x - 6x^2 + 3x - 6 =$

$= x^4 - 2{,}5x^3 - 4x^2 + x - 6$

$\Rightarrow \quad f'(x) = 4x^3 - 7{,}5x^2 - 8x + 1$

Ableiten mit der Produktregel:

$f'(x) = (4x - 1)\cdot(0{,}5x^2 - x - 3) + (2x^2 - x + 2)\cdot(x - 1) =$

$= 2x^3 - 0{,}5x^2 - 4x^2 + x - 12x + 3 + 2x^3 - x^2 + 2x - 2x^2 + x - 2 = 4x^3 - 7{,}5x^2 - 8x + 1$

----

c) $f(x) = (x - 1)\cdot(x + 3) = x^2 - x + 3x - 3 = x^2 + 2x - 3$

$\Rightarrow \quad f'(x) = 2x + 2$

Ableiten mit der Produktregel:

$f'(x) = 1\cdot(x + 3) + (x - 1)\cdot 1 = x + 3 + x - 1 = 2x + 2$

----

d) $f(x) = (x^4 - x^2 + 1)\cdot(x^2 - 2x + 2) =$

$= x^6 - x^4 + x^2 - 2x^5 + 2x^3 - 2x + 2x^4 - 2x^2 + 2 = x^6 - 2x^5 + x^4 + 2x^3 - x^2 - 2x + 2$

$\Rightarrow \quad f'(x) = 6x^5 - 10x^4 + 4x^3 + 6x^2 - 2x - 2$

Ableiten mit der Produktregel:

$f'(x) = (4x^3 - 2x)\cdot(x^2 - 2x + 2) + (x^4 - x^2 + 1)\cdot(2x - 2) =$

$= 4x^5 - 2x^3 - 8x^4 + 4x^2 + 8x^3 - 4x + 2x^5 - 2x^3 + 2x - 2x^4 + 2x^2 - 2 =$
$= 6x^5 - 10x^4 + 4x^3 + 6x^2 - 2$

---

e) $f(x) = (x^3+4x)(x^3+3) = x^6+3x^3+4x^4+12x = x^6+4x^4+3x^3+12x$

$\Rightarrow \quad f'(x) = 6x^5+16x^3+9x^2+12$

Ableiten mit der Produktregel:

$f'(x) = (3x^2+4)\cdot(x^3+3)+(x^3+4x)\cdot 3x^2 = 3x^5+4x^3+9x^2+12+3x^5+12x^3 =$

$= 6x^5+16x^3+9x^2+12$

---

f) $f(x) = (0{,}25x^2-2x+4)\cdot(2x^3+4x-3) =$

$= 0{,}5x^5-4x^4+8x^3+x^3-8x^2+16x-0{,}75x^2+6x-12 =$

$= 0{,}5x^5-4x^4+9x^3-8{,}75x^2+22x-12$

$\Rightarrow \quad f'(x) = 2{,}5x^4-16x^3+27x^2-17{,}5x+22$

Ableitung mit der Produktregel:

$f'(x) = (0{,}5x-2)\cdot(2x^3+4x-3)+(0{,}25x^2-2x+4)\cdot(6x^2+4) =$

$= x^4-4x^3+2x^2-8x-1{,}5x+6+1{,}5x^4-12x^3+24x^2+x^2-8x+16 =$

$= 2{,}5x^4-16x^3+27x^2-17{,}5x+22$

---

4 Quotientenrgel

a) $f(x) = \frac{x}{x+1} \quad \Rightarrow \quad f'(x) = \frac{1\cdot(x+1)\ -\ x\cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}$

---

b) $g(x) = \frac{2x}{1+3x} \quad \Rightarrow \quad g'(x) = \frac{2\cdot(1+3x)\ -\ 2x\cdot 3}{(1+3x)^2} = \frac{2}{(1+3x)^2}$

---

c) $f(z) = \frac{1-z^2}{z+2} \quad \Rightarrow \quad f'(x) = \frac{-2z\cdot(z+2)\ -\ (1-z^2)\cdot 1}{(z+2)^2} = \frac{-z^2-4z-1}{(z+2)^2} = -\frac{z^2+4z+1}{(z+z)^2}$

---

d) $f(t) = \frac{t^2+t+1}{t^2-1} \quad \Rightarrow \quad f'(t) = \frac{(2t+1)\cdot(t^2-1)\ -\ (t^2+t+1)\cdot 2t}{(t^2-1)^2} = -\frac{t^2+4t+1}{(t^2-1)^2}$

---

e) $g(x) = \frac{6x}{15-x^2} \quad \Rightarrow \quad g'(x) = \frac{6\cdot(15-x^2)\ -\ 6x\cdot(-2x)}{(15-x^2)^2} = \frac{90+6x^2}{(15-x^2)^2}$

---

f) $h(z) = \frac{4z^2-5}{2z+1} \Rightarrow h'(x) = \frac{8z\cdot(2z+1) - (4z^2-5)\cdot 2}{(2z+1)^2} = \frac{8z^2+8z+10}{(2z+1)^2} = 2\cdot\frac{4z^2+4z+5}{(2z+1)^2}$

---

5 Differenzieren auf zwei Arten

$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f'(x) = -2x^{-3} = -\frac{2}{x^3}$

Ableitung mit der Quotientenregel:

$f'(x) = \frac{0\cdot x^2 - 1\cdot 2x}{x^4} = -\frac{2}{x^3}$

---

6 Umformungen so, dass die Produktregel anwendbar ist

Vorgegeben :

$g(x) = \sqrt{2x} \Rightarrow g'(x) = \frac{1}{\sqrt{2x}}$

$g(x) = \cos(2x-1) \Rightarrow g'(x) = -2\cdot\sin(2x-1)$

$g(x) = \sin\left(\frac{x}{2}\right) \Rightarrow g'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

$g(x) = \sqrt{1+x^2} \Rightarrow g'(x) = \frac{1}{\sqrt{x^2+1}}$

a) $f(x) = x^2\cdot\sin\left(\frac{x}{2}\right) \Rightarrow f'(x) = 2x\cdot\sin\left(\frac{x}{2}\right) + \frac{1}{2}x^2\cdot\cos\left(\frac{x}{2}\right)$

b) $f(x) = \sqrt{2x}\cdot(x^2-x-1) \Rightarrow f'(x) = \frac{1}{\sqrt{2x}}\cdot(x^2-x-1) + \sqrt{2x}\cdot(2x-1) = \frac{5x^2-3x-1}{\sqrt{2x}}$

c) $f(x) = \sqrt{1+x^2}\cdot\cos(x-1) \Rightarrow f'(x) = \frac{1}{\sqrt{1+x^2}}\cdot\cos(2x-1) - 2\cdot\sqrt{1+x^2}\cdot\sin(2x-1) =$

$= \frac{\cos(2x-1) - 2\cdot(1+x^2)\cdot\sin(2x-1)}{\sqrt{1+x^2}}$

d) $f(x) = \sqrt{2x^3+2x} = \sqrt{2x}\cdot\sqrt{x^2+1} \Rightarrow f'(x) = \frac{1}{\sqrt{2x}}\cdot\sqrt{x^2+1} + \sqrt{2x}\cdot\frac{x}{\sqrt{x^2+1}} =$

$= \frac{x^2+1+2x\cdot x}{\sqrt{2x}\cdot\sqrt{x^2+1}} = \frac{3x^2+1}{\sqrt{2x^3+2x}}$

e) $f(x) = \left[\sin\left(\frac{x}{2}\right)\right]^2 = \sin\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) \Rightarrow f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}\cos\left(\frac{x}{2}\right) =$

$= \sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)$

f) $f(x) = \sqrt{2x^3-4x^2+2x} = \sqrt{2x}\cdot(x-1) \Rightarrow f'(x) = \frac{1}{\sqrt{2x}}\cdot(x-1)+\sqrt{2x} = \frac{3x-1}{\sqrt{2x}}$

---

6 Potenzfunktionen

$f(x) = \frac{1}{x^2} \Rightarrow f'(x) = \frac{-2x}{x^4} = -2x^{-3}$

---

7 Umgehen der Quotientenregel

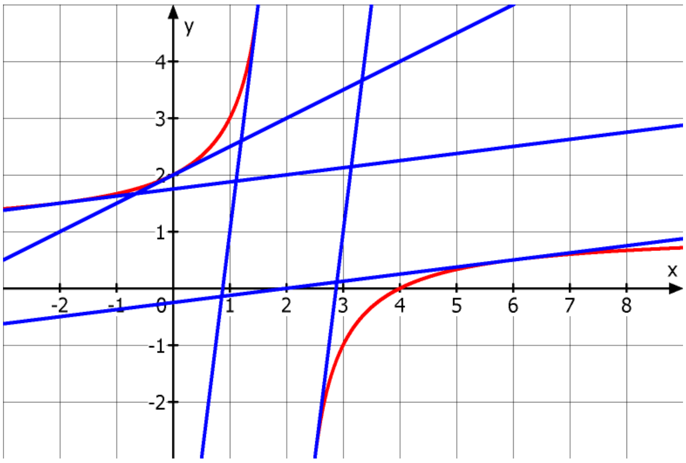
a)	$f(x) = x - \frac{4x^2+1}{2x} = -x - \frac{1}{2x}$	$f'(x) = -1 + \frac{1}{2x^2}$
b)	$f(x) = \frac{3x^2+3x}{x+1} = 3x$	$f'(x) = 3$
c)	$f(x) = \frac{4x^2+1}{2x-1}$	$f'(x) = \frac{8x^2-8x-2}{(2x-1)^2}$
d)	$f(x) = \frac{3-4x^2}{x^3} = \frac{3}{x^3} - \frac{4}{x}$	$f'(x) = -\frac{9}{x^4} + \frac{4}{x^2}$
e)	$f(x) = 2x^3 - \frac{x^2-1}{5}$	$f'(x) = 6x^2 - \frac{2}{5}x$
f)	$f(x) = \frac{2x-3}{2x+3}$	$f'(x) = \frac{12}{(2x+3)^2}$

---

8 Tangenten

$f(x) = \frac{4-x}{2-x} \Rightarrow f'(x) = \frac{-1\cdot(2-x)-(4-x)\cdot(-1)}{(2-x)^2} = \frac{2}{(2-x)^2}$

x	-2	0	1,5	2,5	6
f(x)	1,5	2	5	-3	0,5
f '(x)	1/8	1/2	8	8	1/8



---

9 Berechnung der Ableitung

a) $f(x) = (4x^3 - 2x)\cdot(x^2 + 2x - 3)$

$\Rightarrow \quad f'(x) = (12x^2 - 2)\cdot(x^2 + 2x - 3) + (4x^3 - 2x)\cdot(2x + 2) =$

$= 20x^4 + 32x^3 - 42x^2 - 8x + 6$

---

b) $f(x) = (x^2 - ax + 1)\cdot(x^2 - a) \quad \Rightarrow$

$f'(x) = (2x - a)\cdot(x^2 - a) + (x^2 - ax + 1)\cdot 2x =$

$= 2x^3 - 2ax^2 - 2ax + 2x$

---

c) $f(x) = \dfrac{3a}{1 + x^2} \quad \Rightarrow \quad f'(a) = \dfrac{0\cdot(1 + x^2) - 3a\cdot 2x}{(1 + x^2)^2} = \dfrac{-6ax}{(1 + x^2)^2} = -\dfrac{6ax}{(1 + x^2)^2}$

---

d) $f(a) = \dfrac{3a}{1 + x^2} = \dfrac{3}{1 + x^2}\cdot a \quad \Rightarrow \quad f'(a) = \dfrac{3}{1 + x^2}$

---

e) $g(x) = \dfrac{x^4}{x^4 + 4} \quad \Rightarrow \quad g'(x) = \dfrac{4x^3\cdot(x^4 + 4) - x^4\cdot 4x^3}{(x^4 + 4)^2} = \dfrac{16x^3}{(x^4 + 4)^2}$

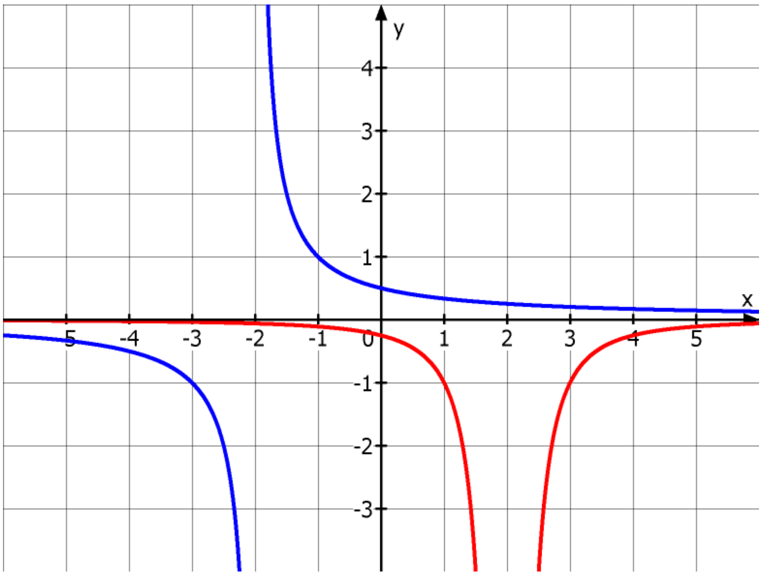
---

f) $h(x) = \dfrac{tx^2 + 2}{x + 1} \quad \Rightarrow \quad f'(x) = \dfrac{2tx\cdot(x + 1) - (tx^2 + 2)\cdot 1}{(x + 1)^2} = \dfrac{tx^2 + 2tx - 2}{(x + 1)^2}$

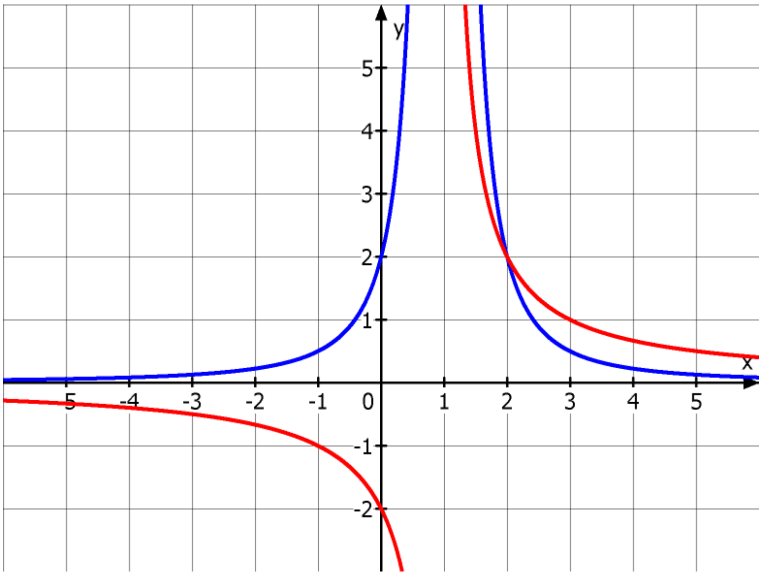
---

10 Graph der Ableitungfunktion bzw. Stammfunktion

a)



b)

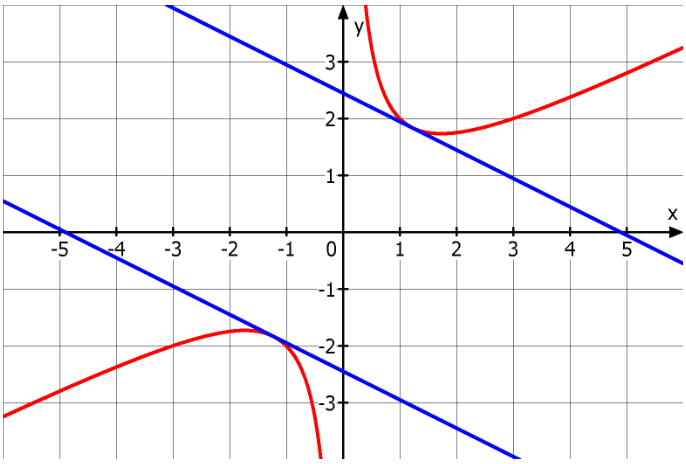


---

11 Bestimmter Ableitungswert

a) $f(x) = \frac{x}{2} + \frac{3}{2x} \quad \Rightarrow \quad f'(x) = \frac{1}{2} - \frac{3}{2x^2}$

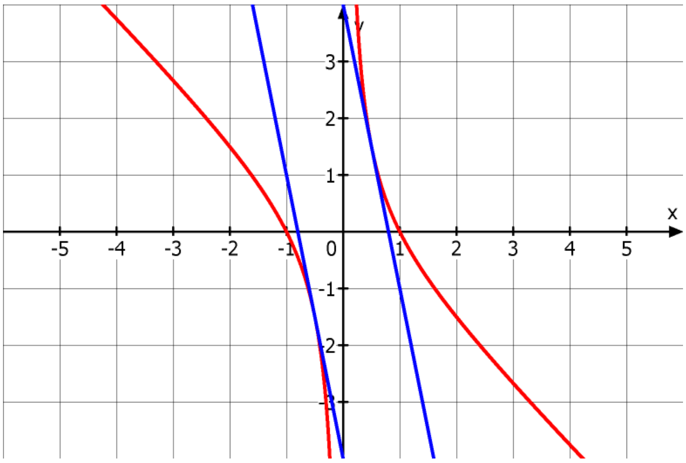
Bedingung : $\frac{1}{2} - \frac{3}{2x^2} = -\frac{1}{2} \quad \Rightarrow \quad \frac{3}{2x^2} = 1 \quad \Rightarrow \quad x = -\sqrt{\frac{3}{2}} \vee x = \sqrt{\frac{3}{2}}$



---

b) $f(x) = \frac{1-x^2}{x} = \frac{1}{x} - x \quad \Rightarrow \quad f'(x) = -\frac{1}{x^2} - 1$

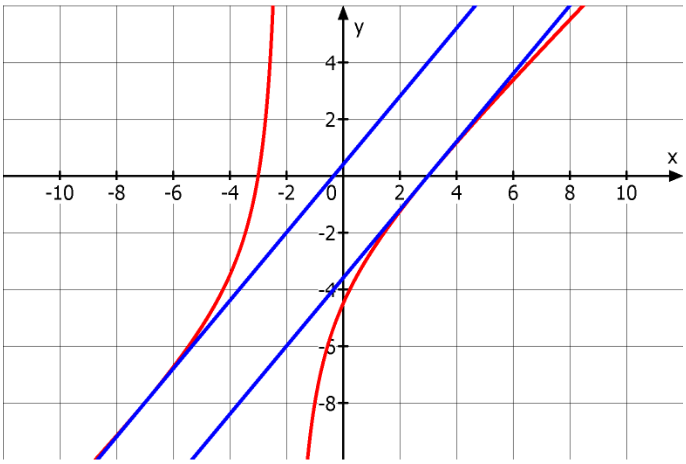
Bedingung : $-\frac{1}{x^2} - 1 = -5 \quad \Rightarrow \quad x = -\frac{1}{2} \vee x = \frac{1}{2}$



---

c) $f(x) = \frac{x^2-9}{x+2} \quad \Rightarrow \quad f'(x) = \frac{2x \cdot (x+2) - (x^2-9) \cdot 1}{(x+2)^2} = \frac{x^2+4x+9}{(x+2)^2}$

Bedingung : $\frac{x^2+4x+9}{(x+2)^2} = \frac{6}{5} \quad \Rightarrow \quad x = -7 \vee x = 3$

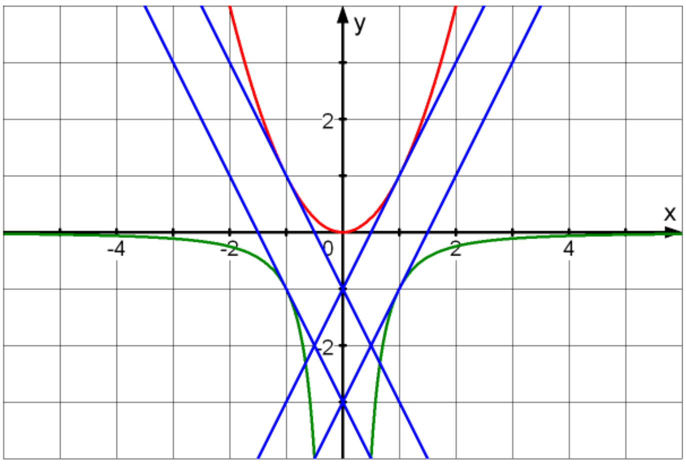


---

12 Parallele Tangenten

a) $f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x \qquad g(x) = -\frac{1}{x^2} \quad \Rightarrow \quad g'(x) = \frac{2}{x^3}$

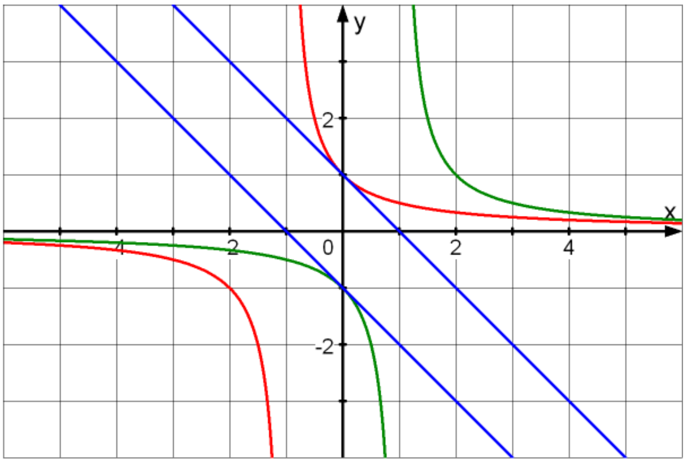
$$2x = \frac{2}{x^3} \quad \Rightarrow \quad x = -1 \lor x = 1$$



---

b) $f(x) = \frac{1}{x+1} \quad \Rightarrow \quad f'(x) = -\frac{1}{/x+1)^2} \qquad g(x) = \frac{1}{x-1} \quad \Rightarrow \quad g'(x) = -\frac{1}{(x-1)^2}$

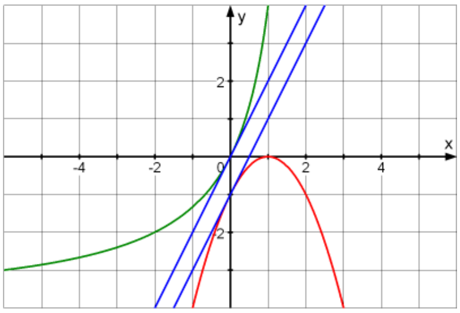
$$\frac{1}{(x+1)^2} = \frac{1}{(x-1)^2} \quad \Rightarrow \quad x = 0$$



c) $f(x) = -x^2 + 2x - 1 \quad \Rightarrow \quad f'(x) = -2x + 2$

$$g(x) = \frac{-4x}{x-2} \quad \Rightarrow \quad g'(x) = \frac{8}{(x-2)^2}$$

$$-2x + 2 = \frac{8}{(x-2)^2} \quad \Rightarrow \quad x = 0$$



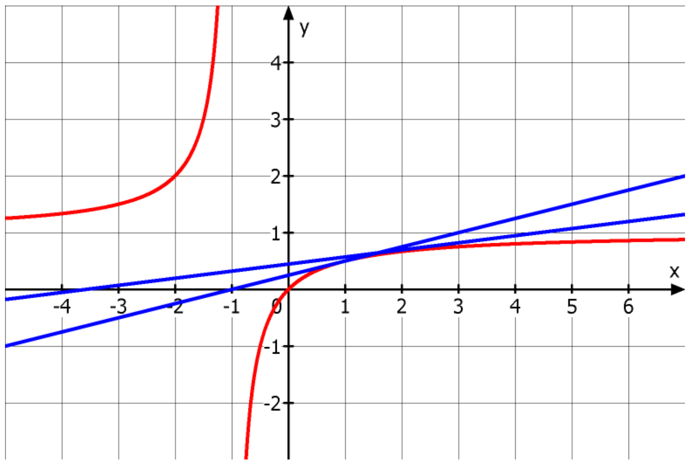
---

13 Gleichungen von Tangenten

a) $f(x) = \dfrac{x}{x+1} \quad \Rightarrow \quad f'(x) = \dfrac{1}{(x+1)^2}$

In $P\left(1 | 0{,}5\right): y = \frac{1}{4} \cdot (x-1) + \frac{1}{2} = \frac{1}{4}x + \frac{1}{4}$

In $P\left(2 | \frac{2}{3}\right): y = \frac{1}{9} \cdot (x-2) + \frac{2}{3} = \frac{1}{9}x + \frac{4}{9}$

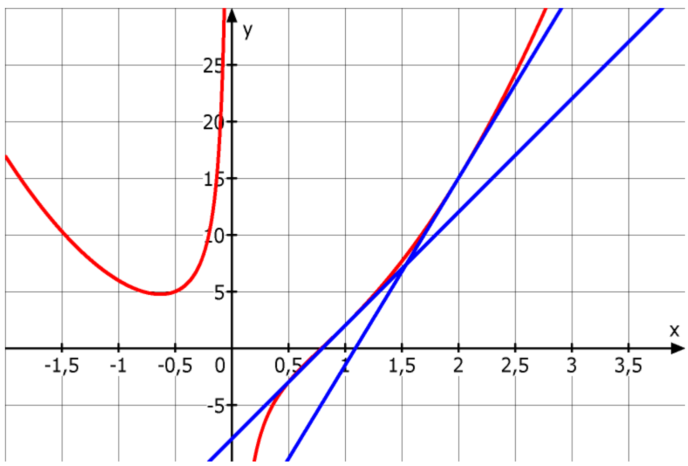


---

b) $f(x) = \frac{4x^3 - 2}{x} \quad \Rightarrow \quad f'(x) = 8x + \frac{2}{x^2}$

In $P\left(1|\, 2\right): y = 10\cdot(x-1) + 2 = 10x - 8$

In $P\left(2|\, 15\right): y = \frac{33}{2}\cdot(x-2) + 18 = \frac{33}{2}x - 15$



---

14 Tangenten

a) $f(x) = \frac{2x}{x-2} \quad \Rightarrow \quad f'(x) = \frac{2\cdot(x-2) - 2x\cdot 1}{(x-2)^2} = \frac{-4}{(x-2)^2}$

Allgemeine Tangentengleichung im Berührpunkt $B(x_0 \mid \frac{2x_0}{x_0 - 2})$-:

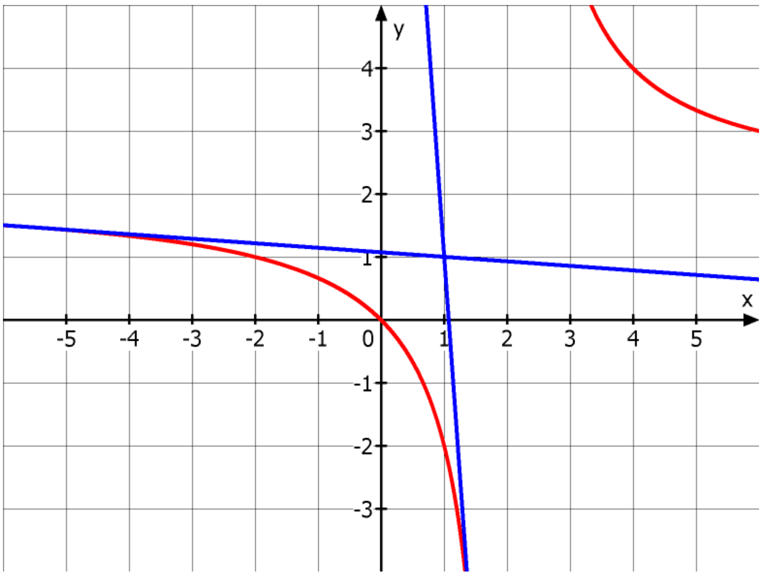
$$y = -\frac{4}{(x_0-2)^2}\cdot(x - x_0) + \frac{2x_0}{x_0 - 2}$$

$P\left(1 \mid 1\right)$ eingesetzt:

$$1 = -\frac{4}{(x_0-2)^2}\cdot(1-x_0) + \frac{2x_0}{x_0-2} \quad\Leftrightarrow\quad (x_0-2)^2 = -4\cdot(1-x_0) + 2x_0\cdot(x_0-2)$$

$$\Leftrightarrow \quad x = -2-2\sqrt{3} \vee x = -2+2\sqrt{3}$$

$$y = -\frac{4}{(-4-2\sqrt{3})^2}\cdot(x-1)+1 \text{ bzw. } y = -\frac{4}{(-4+2\sqrt{3})^2}\cdot(x-1)+1$$



b) $f(x) = \frac{2x+1}{2x-1} \quad\Rightarrow\quad f'(x) = \frac{2\cdot(2x-1) - (2x+1)\cdot 2}{(2x-1)^2} = \frac{-4}{(2x-1)^2}$

Allgemeine Tangentengleichung im Berührpunkt $B(x_0 \mid \frac{2x_0+1}{2x_0-1})$-:

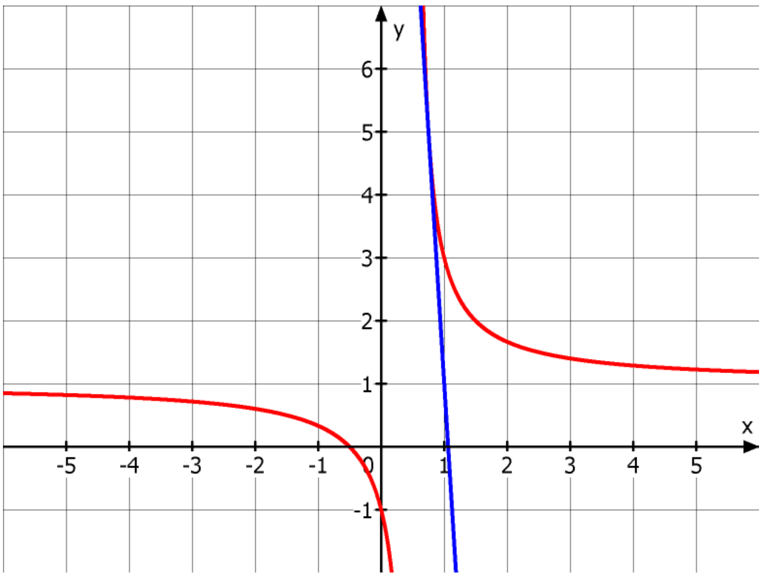
$$y = -\frac{4}{(2x_0-1)^2}\cdot(x-x_0) + \frac{2x_0+1}{2x_0-1}$$

$P\left(1 \mid 1\right)$ eingesetzt:

$$1 = -\frac{4}{(2x_0-1)^2}\cdot(1-x_0) + \frac{2x_0+1}{2x_0-1} \quad\Leftrightarrow\quad (2x_0-1)^2 = -4\cdot(1-x_0) + (2x_0+1)\cdot(2x_0-1)$$

$$\Leftrightarrow \quad x = \frac{3}{4}$$

$$y = -16\cdot(x-1)+1$$



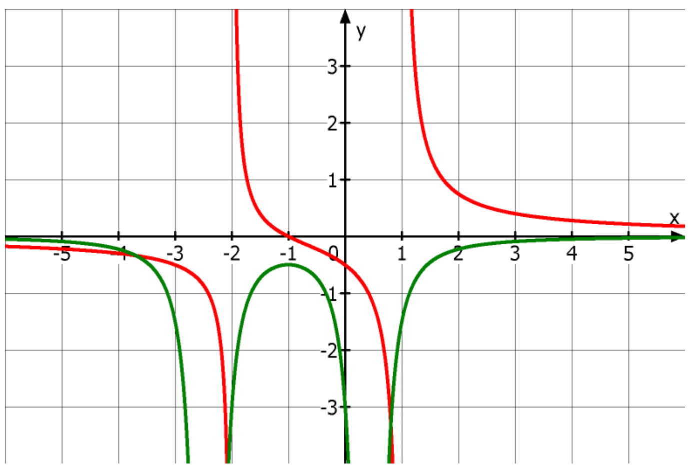
---

15 Zuordnung des Graphen einer Funktion zu deren Ableitung

Es kann sich nur um den rechts unten eingezeichneten Graphen handeln.

$$f(x) = \frac{x+1}{(x+2)\cdot(x-1)} = \frac{x+1}{x^2+x-2} \quad \Rightarrow \quad f'(x) = \frac{1\cdot(x^2+x-2)-(x+1)\cdot(2x+1)}{(x^2+x-2)^2} =$$

$$= \frac{-x^2-2x-3}{(x^2+x-3)^2}$$



---

16 Konzentration eines Medikaments

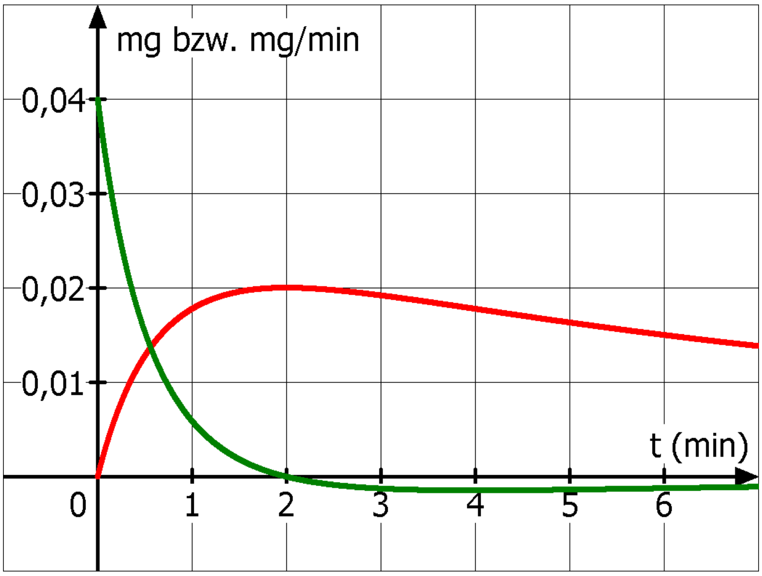
a) $f(t) = 0{,}16\cdot\frac{t}{(t+2)^2} \quad \Rightarrow \quad f'(t) = 0{,}16\cdot\frac{1\cdot(t+2)^2-t\cdot(2t+2)}{(t+2)^4} =$

$$= 0{,}16\cdot\frac{t+2-2t}{(t+2)^3} = 0{,}16\cdot\frac{2-t}{(t+2)^3}$$

$$f'(3) = -0{,}0064\left(\frac{mg}{min}\right) \quad f'(6) = -0{,}00125\left(\frac{mg}{min}\right)$$

Mittlere Änderungsrate im Zeitintervall $\left[0\text{ min}; 6\text{ min}\right]$-.

$$\frac{f(6)-f(0)}{6} = \frac{0{,}015}{6} = 0{,}0025\left(\frac{mg}{min}\right)$$



---

17 Ordnung der Pole

$$f(x) = \frac{g(x)}{(x-a)^n} \text{ mit } g(a) \neq 0$$

$$f'(x) = \frac{g'(x)\cdot(x-a)^n - g(x)\cdot n\cdot(x-a)^{n-1}}{(x-a)^{2n}} = \frac{g'(x)\cdot(x-a) - g(x)}{(x-a)^{n+1}}$$

und $z(a) = g'(a)\cdot(a-a) - g(a) = g(a) \neq 0$

G 18 Geradengleichung

$A(4|\,2)$ und $B\left(-5|\,\frac{1}{3}\right)$:

$$m = \frac{\frac{1}{3}-2}{-5-4} = \frac{-\frac{5}{3}}{-9} = \frac{5}{27}$$

$$y = \frac{5}{27} \cdot (x - 4) + 2 = \frac{5}{27} x + \frac{34}{27}$$
