Magic circles in the arbelos

by

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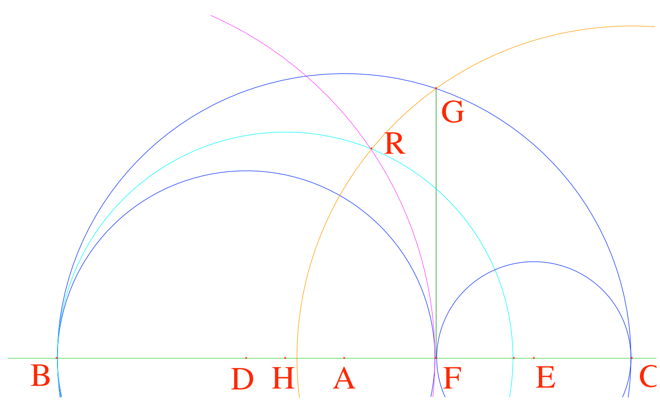
Abstract. In the *arbelos* three simple circles are constructed on which the tangency points for three circle chains, all with Archimedes' circle as a common starting point, are situated. In relation to this setting, some algebraic formulae and remarks are presented.

Key words: Geometry, circle chains, arbelos, inversion

Mathematical Subject Classification: Primary: 51N20

Introduction

The *arbelos*[1], or 'the shoemaker's knife', studied already by Archimedes, is a configuration of three tangent circles, and may indeed, due to its many remarkable properties, be called three magic circles. In this paper the focus will be on three circles linked to the *arbelos*, involved in the construction of Archimedes' circles[2] and chains of tangent circles. These three circles, which I also want to call 'magic', are displayed in the figure below. The *arbelos* is conctructed by the circles with diameters BC, BF, and FC, respectively, and FG is the perpendicular to BC. The magic circles I will discuss in this paper are the circles with centre at B passing F, with centre at C passing G, and with centre on BC (at H) passing B and the intersection point R of the previous two magic circles. A parallel "mirror" construction of these circles can be performed on the right side of the segment FG.



As one starting point of my discussion, I will look at the following basic construction. One aspect of Archimedes' "twin" circles in the *arbelos* involves constructing a circle tangent to

[1] See for example [3]

[2] See for example [3].

a given circle and its tangent. Given the point of tangency on the given circle this is easily done as in figure 1.

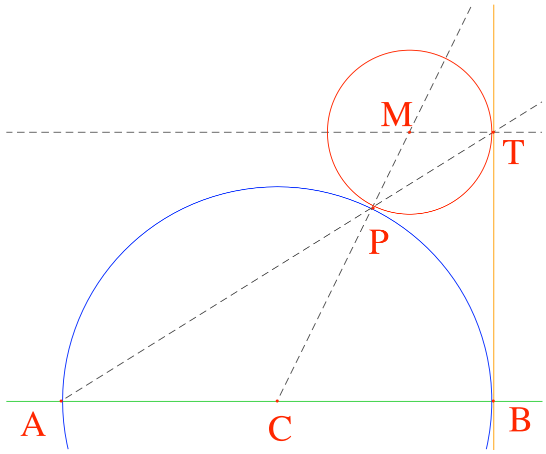


Figure 1. Construction of tangent circle

A circle on diameter AB with centre C is given[3], as well as a point P on this circle, and the line tangent to the circle at B. Draw the line by A and P, meeting the tangent in T. The perpendicular to the tangent by T meets the line by C and P at M. Thus, the tangent circle is found with centre at M and tangency points P and T. The construction is built on the classic result that the segment connecting opposite endpoints of parallel diameters to externally tangent circles passes the point of tangency. That MP and MT are equal follows from the similarity of the triangles ACP and TMP, AC and TM being parallel.

Magic circle 1

It is evident that the circle $C_{A,B}$ meets any such tangent circle at a right angle. To prove this, apply circle inversion of these three circles and the tangent line to the circle $C_{B,A}$, as in figure 2.

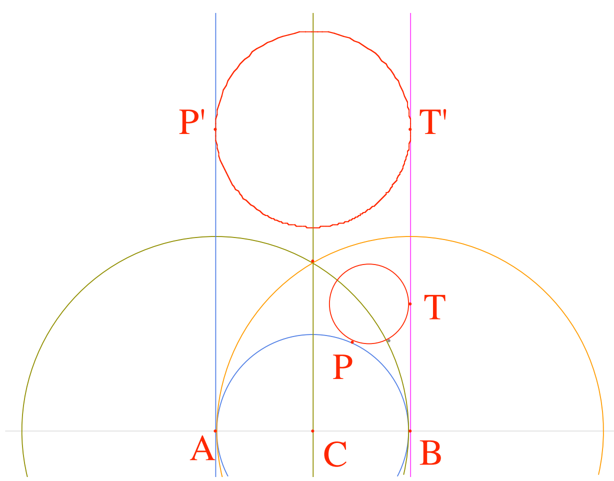


Figure 2. Inversion in circle with centre at B.

[3] For this paper I will use the notation $C_{C,B}$ for a circle with centre at the point C with radius CB.

As point B is mapped, by this inversion, to infinity, the circle $C_{C,B}$ is mapped to line through A and P', and circle $C_{A,B}$ to the perpendicular to AB at C. P and T are mapped to P' and T', respectively, and the circle with tangent points P and T to the circle tangent to the lines by A and P', and T and T', respectively, thus orthogonal to the perpendicular to AB at C. Because of this orthogonality property I call the circle $C_{A,B}$ a *magic circle*. How this circle is related to a circle chain in the *arbelos* will be discussed in the next section.

Magic circle 2

Given its tangency points on the inner circles of the *arbelos*, the construction in figure 1 can be used to construct Archimedes' circles. A simple construction is shown in figure 3, where the circle $C_{C,G}$ meets the left inner circle at point P, which is a tangency point of the left twin circle. Because of this I call the circle $C_{C,G}$ a magic circle. The other two tangency points, P' and P'', are found by drawing the lines by B and P, and by C and P', respectively (see figure 3).

That P is a tangency point can be established algebraically, using the centre coordinates $\left(\frac{r}{2}(1+r), r\sqrt{1-r}\right)$ of the left Archimedes' circle[4] to find the coordinates of its tangency point $\left(\frac{r}{2-r}, \frac{r\sqrt{1-r}}{2-r}\right)$, which has the same distance $\sqrt{1-r}$ to the point C as the point G.

In addition to the properties shown in figure 1, the diameter at P' to the circle passing P, P', and P'', is parallel to the diameter BC of the circle $C_{A,C}$, showing that P'' is the point of tangency between these two circles as it is constructed by the line by CP', i.e. passing the end points of the parallel diameters.

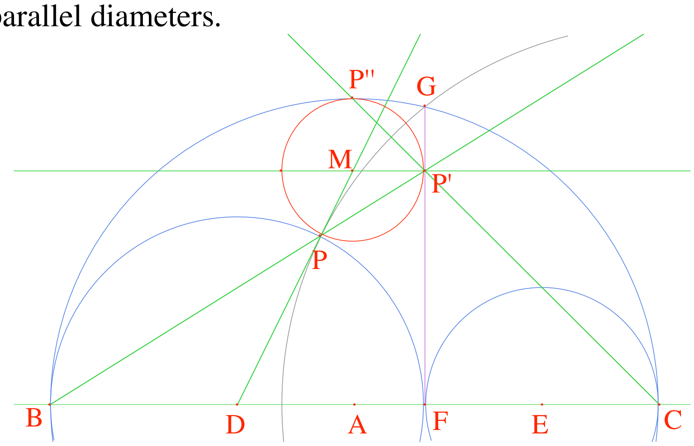


Figure 3. A construction of the left twin circle.

[4] Here, BC = 1 and BF = r, with $0 < r < 1$. See for example [3].

The centre M of the twin circle is found as in figure 1, where the perpendicular line to FG at P’ meets the line by D and P (for this construction the point P’’ is not needed). The construction of the right twin circle is done the same way, by first drawing the circle $C_{B,G}$.

Applying the result from the inversion described in figure 2, chains of tangent circles as shown in figure 4 are easily constructed by drawing the magic circle $C_{B,F}$ (and $C_{C,F}$) in the *arbelos* (with notations as in figure 3)[5].

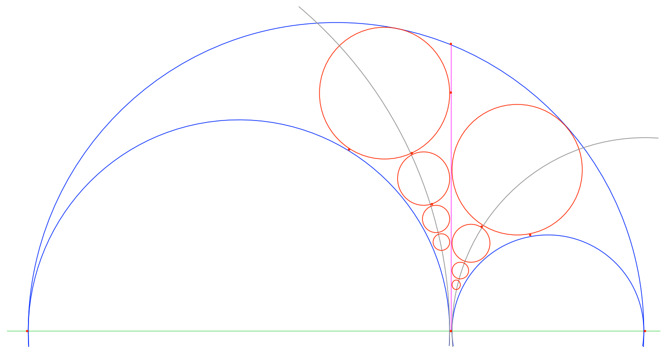


Figure 4. Circle chains in the arbelos

The mode of construction is the same as in figure 1, illustrated in figure 5, where the first circle in the chain after the left twin circle is shown. The inversion in figure 2 proves the tangency of this circle also to the left inner circle of the *arbelos*.

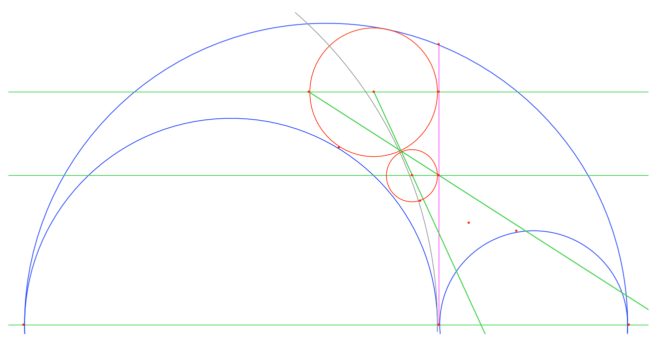


Figure 5. Construction of first tangent circle in chain

The radii of these chain circles can be evaluated (by algebraic calculation using Pythagoras’ theorem on the marked triangles in figure 6) by the following recursion formula:

$r_{n+1} = \dfrac{r \cdot r_n}{\left(\sqrt{r} + \sqrt{2r_n}\right)^2}$, $n = 1,2,3,\ldots$, where $r_1 = \dfrac{r}{2}(1-r)$ is the radius for Archimedes’ circle, as well known. By a straightforward mathematical induction, the explicit formula

[5] This magic circle $C_{B,F}$ is an example of a Woo circle (see [2])

$r_n = \dfrac{r(1-r)}{2\left(1+(n-1)\sqrt{1-r}\right)^2}$, $n = 1,2,3,\ldots$ can be established. A calculation of the coordinates (x_n, y_n) of the circle centres gives

$\begin{cases} x_n = r - r_n \\ y_n = \sqrt{2r \cdot r_n} \end{cases}, n = 1,2,3,\ldots$, where the y-coordinate can be simplified to

$y_n = \dfrac{r\sqrt{1-r}}{1+(n-1)\sqrt{1-r}}$, $n = 1,2,3,\ldots$

Obviously, $r_n \to 0$ and $(x_n, y_n) \to (r,0)$, and also $\dfrac{r_{n+1}}{r_n} \to 1$ as $n \to \infty$.

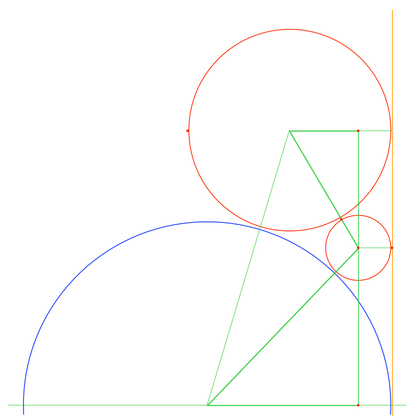


Figure 6. Triangles used for establishing the recursion formula.

By symmetry, the corresponding formulæ for the tangent circle chain to the right Archimedean circle, with radii r'_n , are

$r'_{n+1} = \dfrac{(1-r)\cdot r'_n}{\left(\sqrt{1-r}+\sqrt{2r'_n}\right)^2}$, $n = 1,2,3,\ldots$, and $r'_n = \dfrac{r(1-r)}{2\left(1+(n-1)\sqrt{r}\right)^2}$, $n = 1,2,3,\ldots$,

and the centre coordinates $\begin{cases} x'_n = r + r'_n \\ y'_n = \sqrt{2(1-r)\cdot r'_n} \end{cases}, n = 1,2,3,\ldots$,

where the y-coordinate can be simplified to $y'_n = \dfrac{(1-r)\sqrt{r}}{1+(n-1)\sqrt{r}}$, $n = 1,2,3,\ldots$

The circle $C_{C,G}$ is double magic because it can be used to construct both the twin circle and the chain of tangent circles shown in figure 7.

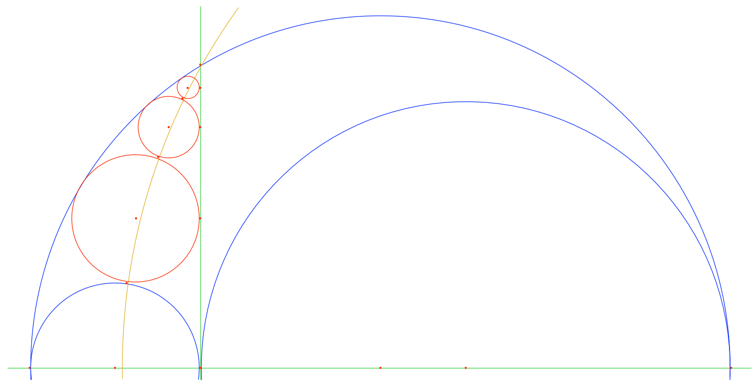


Figure 7. A chain of tangent circles in the *arbelos*.

In figure 7 the radius of the left inner circle of the *arbelos* has been made small to make it possible to visualise the chain of tangent circles having their points of tangency on the circle $C_{C,G}$ (see notation in figure 3). To prove this tangency property I note that the tangent *t* to circle $C_{C,G}$ is the angle bisector to the line FG and the tangent *t'* to circle $C_{A,C}$ (see figure 8, where P is a point on t' left to G). Using the notations

$$\begin{cases} \alpha = \wedge BGF \\ \beta = \wedge BGP \\ \varphi = \wedge AGF \end{cases}$$

an algebraic calculation shows that $\tan\alpha = \tan\beta$.

Indeed, $\tan\alpha = \dfrac{r}{\sqrt{r(1-r)}}$, $\tan\varphi = \dfrac{r-\frac{1}{2}}{\sqrt{r(1-r)}}$, and $\beta + (\alpha - \varphi) = \dfrac{\pi}{2}$. Therefore

$\tan\beta = \dfrac{1}{\tan(\alpha-\varphi)} = \dfrac{1+\tan\alpha\cdot\tan\varphi}{\tan\alpha - \tan\varphi} = \dfrac{r}{\sqrt{r(1-r)}}$, after simplification.

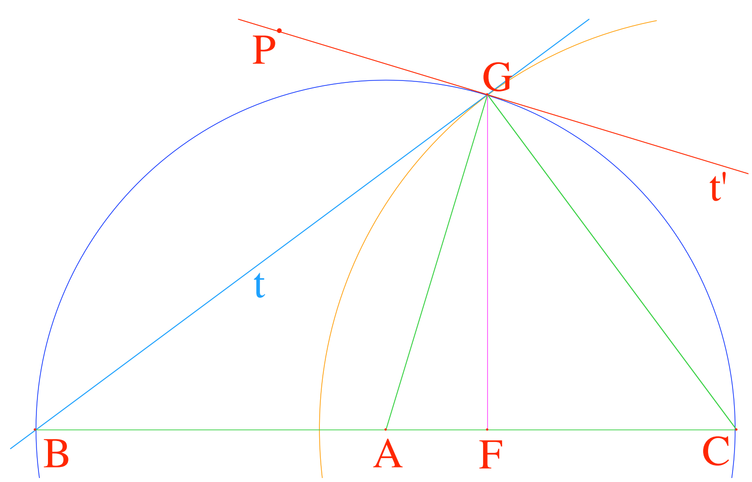


Figure 8. Angle bisector.

The tangency property to be proved now follows by circle inversion of the line by FG and the circles $C_{A,C}$ and $C_{C,G}$ in the circle $C_{G,F}$. Since the point G by this inversion is mapped to infinity the line by FG and the circles $C_{A,C}$ and $C_{C,G}$ all are mapped on lines, intersecting at the mid point on FG with the image of the circle $C_{C,G}$ being the bisector to the other two lines. This proves the tangency property, illustrated in figure 9 (where r is small to make it possible to visualise the inversion), where Archimedes' circle and one more circle in the chain and its inverses are shown. The left inner circle of the *arbelos* is mapped onto itself, and the line by the diameter BC onto the circle with diameter FG.

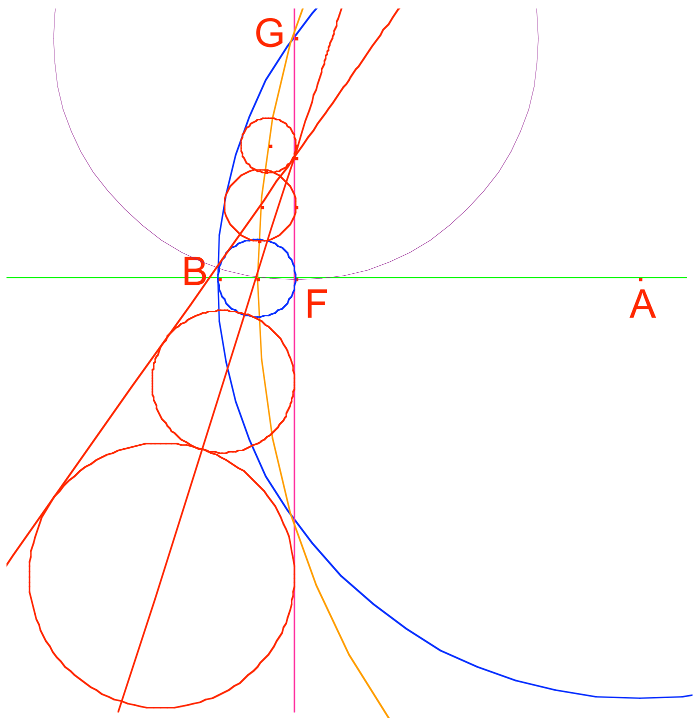


Figure 9. Inversion of circle chain.

By elementary geometry, using algebra, a recursion formula for the radii of the circle chain in figure 7 can be established. Setting the Archimedean circle as the first circle in this chain, the result of this computation is the formula

$$r_{n+1} = \frac{r_n(1-r)}{\left(1+\sqrt{r-2r_n}\right)^2} \text{, for } n = 1,2,3,\ldots \text{ with } r_1 = \frac{r}{2}(1-r)\ .$$

By this recursion formula the following explicit formula can be proved by mathematical induction[6]:

[6] I'm grateful to Thomas Bäckdahl for finding the explicit formula for the polynomial P_n from the number pattern of the coefficients of the first ten polynomials.

$r_n = \frac{r}{2} \cdot \frac{(1-r)^n}{P_n^{\,2}(r)}$, $n = 1,2,3,\ldots,$ where $P_n(r) = \frac{1}{2}\left(\left(1+\sqrt{r}\right)^n + \left(1-\sqrt{r}\right)^n\right)$, which is a polynomial in r of degree $\left[n/2\right]$. For example, for $n = 2$ and $n = 5$ these radii are

$$r_2 = \frac{r}{2} \cdot \frac{(1-r)^2}{(1+r)^2} \quad \text{and} \quad r_5 = \frac{r}{2} \cdot \frac{(1-r)^5}{(1+10r+5r^2)^2} .$$

To complete the proof by induction the equality $\frac{r_n(1-r)}{\left(1+\sqrt{r-2r_n}\right)^2} = \frac{r}{2} \cdot \frac{(1-r)^{n+1}}{P_{n+1}^{\,2}(r)}$ must be established with $r_n = \frac{r}{2} \cdot \frac{(1-r)^n}{P_n^{\,2}(r)}$, which means that the equality $\sqrt{rP_n^2(r) - r(1-r)^2} = P_{n+1}(r) - P_n(r)$ must hold. This is easily seen, since the right hand side equals $\frac{\sqrt{r}}{2}\left(\left(1+\sqrt{r}\right)^n - \left(1-\sqrt{r}\right)^n\right)$, and

$$P_n^2(r) - (1-r)^2 = \frac{1}{4}\left(\left(1+\sqrt{r}\right)^{2n} + \left(1-\sqrt{r}\right)^{2n} + 2(1-r)^n\right) - (1-r)^n = \left(\frac{\left(1+\sqrt{r}\right)^n - \left(1-\sqrt{r}\right)^n}{2}\right)^2$$

It is obvious that for $r > \frac{1}{2}$ these circles are very small. For example, $r = \frac{2}{3}$ gives $r_2 = \frac{1}{75}$ and already $r_3 = \frac{1}{729}$, as compared to the diameter BC = 1. From the recursion formula it follows directly by mathematical induction that $r_n < \frac{r}{2} \cdot (1-r)^n$, and therefore that for large n the quotient $\frac{r_{n+1}}{r_n} < \frac{(1-r)}{\left(1+\frac{1}{2}\sqrt{r}\right)^2}$, which means that the radius is decreasing more rapidly than by a geometric progression.

The corresponding circle centres (x_n, y_n) can be expressed by r and n by using the expression for r_n above and the geometrically derived relations

$$\begin{cases} x_n = r - r_n \\ y_n^{\,2} = (1-r)(r-2r_n) \end{cases}$$

It is obvious that these points indeed approach the point G (see figure 9) as $n \to \infty$.

Magic circle 3

The circles $C_{B,F}$ and $C_{C,G}$ meet in R. Draw the perpendicular bisector to BR, meeting BC in H. Draw the circle $C_{H,R}$. This circle is orthogonal to any circle tangent to both circles $C_{A,B}$ and $C_{D,B}$, and I therefore call it a magic circle. To prove this orthogonality property, use circle inversion in the circle $C_{B,F}$ (see figure 10).

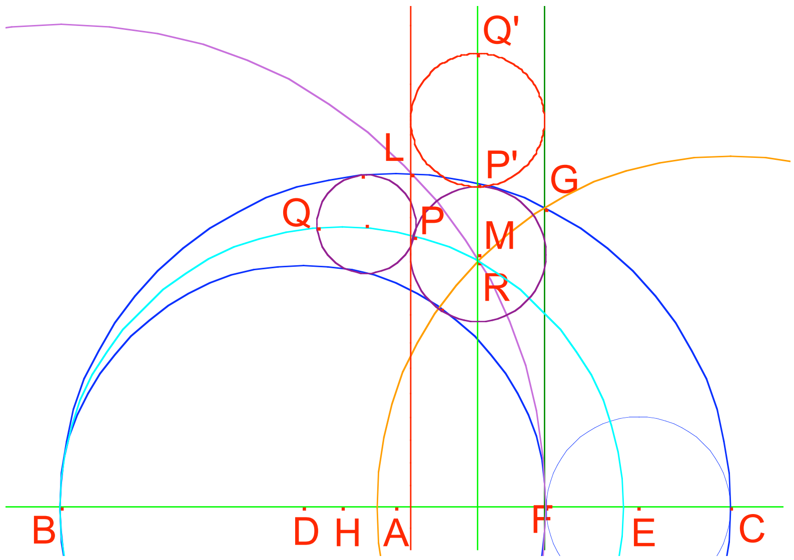


Figure 10. Inversion in circle $C_{B,F}$.

Since the point B is mapped to the infinity point by this inversion, the circle $C_{D,B}$ is mapped to the line by FG (and vice versa), and the circle $C_{A,B}$ is mapped to the line perpendicular to BC passing L (the point of intersection between the circles $C_{A,B}$ and $C_{B,F}$), and the circle $C_{H,R}$ to the line perpendicular to BC passing the point R of intersection between the circles $C_{H,R}$ and $C_{B,F}$. The *x*-coordinate of R is easily calculated to $\frac{r}{2}(1+r)$, and of L to r^2 , showing that Archimedes' circle is tangent to the line by L perpendicular to BC, having its centre on the line by R perpendicular to BC. This also proves the assertion above, illustrated in figure 10 by the points P and Q and their inverted points P' and Q'. The twin circle is mapped onto itself.

The radius of the magic circle $C_{H,R}$ is by elementary algebra found to be $\frac{r}{1+r}$, which is the harmonic mean of the radii of the circles $C_{D,F}$ and $C_{A,C}$. As known, the harmonic mean is also connected to Archimedes' circles: the diameter of the Archimedean circle is the harmonic mean of the radii of the circles $C_{D,F}$ and $C_{E,F}$.

I also note that the magic circle $C_{H,R}$ meets the circle $C_{E,C}$ at a tangency point of the incircle of the *arbelos*, and so does the corresponding magic circle with a radius that equals the harmonic mean of the circles $C_{E,C}$ and $C_{A,C}$ meet the other tangency point of the incricle (see figure 11). This is a direct consequence of the orthogonality property of these magic circles I discussed above, and provides one way of constructing this incircle[7]. The circle $C_{H,R}$ is thus also double magic. It can also be observed that the line passing the two intersection points of the magic circles $C_{H,R} = C_{H,B}$ and $C_{H',C}$ passes the centre of the *arbelos* incircle. Using the radius $HB = \dfrac{r}{1+r}$ and the radius $H'C = \dfrac{1-r}{2-r}$, the *x*-coordinate of these intersection points can be algebraically determined to be $\dfrac{r(1+r)}{2\left(1-r+r^2\right)}$, which equals the *x*-coordinate of the centre of the incircle (see [1]).

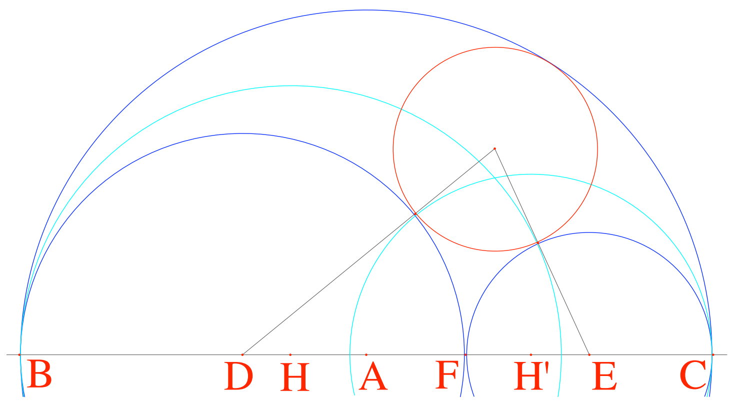


Figure 11. Magic circles and the *arbelos* incircle.

An algebraic calculation, using Pythagoras´ theorem and the length of the radius HB, shows that the centres (x_n, y_n), $n = 1, 2, 3, \ldots$ of the circles in the circle chain orthogonal to $C_{H,R}$ and tangent to the circles $C_{D,F}$ and $C_{A,C}$ can be expressed by their radii r_n with the formulae

$$\begin{cases} x_n = r_n \cdot \dfrac{1+r}{1-r} \\ {y_n}^2 = r \cdot \dfrac{2r_n}{1-r} \cdot \left(1 - \dfrac{2r_n}{1-r}\right) \end{cases}$$

[7] See for example [4] for some simple constructions of the *arbelos* incircle.

where the circle for $n=1$ may be, for example, the incircle of the *arbelos*, or the left Archimedean circle[8]. With this latter choice of $r_1 = \frac{r}{2}(1-r)$ an algebraic computation yields that $r_2 = \frac{(1-r)\left(2-2r+r^2 \pm 2(1-r)\sqrt{1-r}\right)}{2\left(4-4r+r^3\right)}$, where the sign is related to which side of the Archimedean circle the tangent circle is located. The general explicit formula for the radius r_n seems to be complex[9]. It can also be observed that for $r=\frac{3}{4}$, the right-sided tangent circle to Archimedes' circle is the *arbelos* incircle, then having the radius $\frac{3}{26}$, and the magic circle $C_{C,G}$ then having the same radius as the basic circle $C_{A,C}$ (see figure 12).

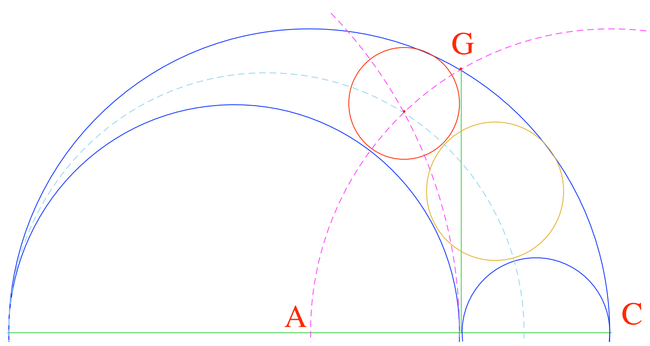


Figure 12. Magic circles with *arbelos* incircle tangent to Archimedes' circle.

For the construction of this circle chain, draw the line by R and Q (as in figure 10), and draw the line by A parallel to RQ, meeting the circle $C_{A,C}$ in P (see figure 13). The line by P and Q meets the circle $C_{A,C}$ in P', which then must be a tangency point for the next circle in the chain. Therefore, the intersection point of the radius AP' and RQ produced will meet in the centre point N of the tangent circle, which thus can be drawn.

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[8] It is easily checked that these circle centres (x_n, y_n), for any choice of r_n, lie on the ellipse

$$\frac{\left(x-\frac{1+r}{4}\right)^2}{\left(\frac{1+r}{4}\right)^2}+\frac{y^2}{\left(\frac{\sqrt{r}}{2}\right)^2}=1$$ (most often discussed in relation to *Pappus chain*; see [1])

[9] In the case with *Pappus chain*, where r_1 is the radius of the *arbelos* incircle, the formula is known to be $r_n = \frac{r(1-r)}{2\left(r+n^2(1-r)^2\right)}$. See [1].

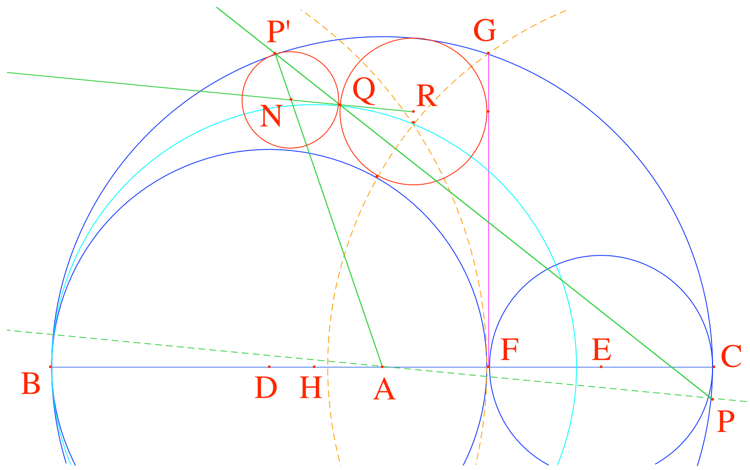

Figure 13. Construction of circle chain.

Another observation points to the fact that the line by R and R', meeting FG at U, is perpendicular to the segment AU (see figure 14), since R and R' are equidistant from FG (as shown above) and the segments AR and AR' both are equal to $\frac{1}{2}\sqrt{r^2+(1-r)^2}$. This means that the segment QQ' is divided into two equal parts by the segment FG.

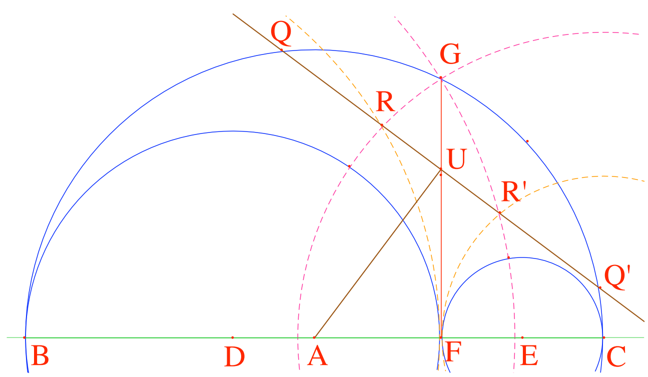

Figure 14. Segments RR' and AU are perpendiclar.

I also note the circle $C_{A,R}$ passes the midpoints of the circle arcs BF and FC, respectively, and has the same area as the circle passing the midpoint of the arc BC, and the points B and F (see figure 15). These observations are established by simple algebraic calculations.

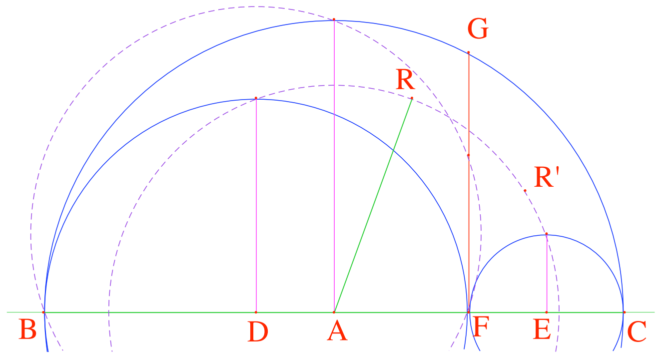

Figure 15. Two equal circles by arc midpoints.

A simple algebraic calculation also shows that the circle ring between the circles $C_{B,G}$ and $C_{B,F}$ has the same area as the circle ring between the circles $C_{C,G}$ and $C_{C,F}$ (see figure 13), equal to $\pi r(1-r)$, which equals the area of the circle with radius FG.

The magic circles and circle chains – a summary

In figure 16 the three magic circles and the circle chains related to them are shown. Their constructions and some of their characteristics, including some algebraic representations, have been presented above. Similarly, the corresponding circles can be constructed on the right side of the perpendicular segment in the *arbelos*. Archimedes' circle thus serves as the starting circle of three different circle chains in the *arbelos*, associated with each of the three magic circles. It has been shown above that the intersection point of these three circles and the centre of Archimedes' circle lie on the same perpendicular line to the common diameter of the *arbelos* circles. It has also been observed that that two of the magic circles meet the inner circles of the *arbelos* at the tangency points of the *arbelos* incircle. Some additional observations have also been presented.

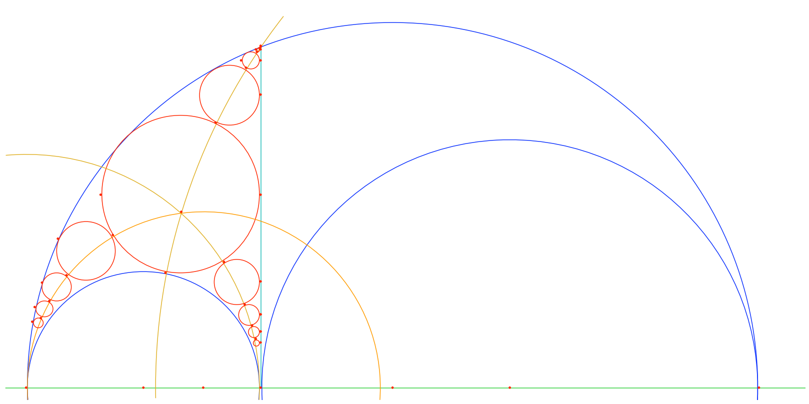

Figure 16. Magic circles and circle chains.

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