The Geometry of The Arbelos

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April, 1998

The basic figure of the arbelos (meaning "shoemaker's knife") is formed by three semicircles with diameters on the same line.

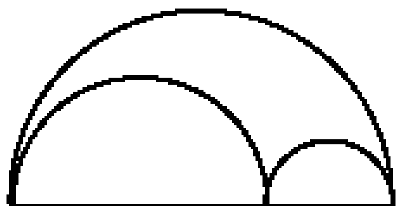

Figure 1. The Shoemaker's Knife

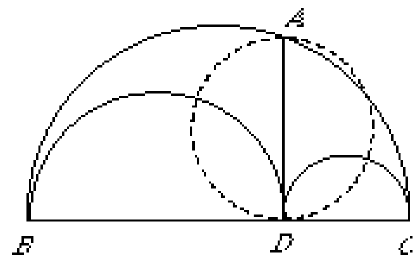

Figure 2.

Archimedes (died 212 BC) included several theorems about the arbelos in his "Book of Lemmas". For example, in Figure 2, the area of the arbelos is equal to the area of the circle with diameter AD. The key is that $AD^2 = BD \cdot DC$ then expand $BC^2 = (BD + DC)^2$.

Another result is that, in Figure 3, the circles centred at E and F have the same radius. The two drawings of the figure illustrate how the theorem does not depend on the position of D on the base line.

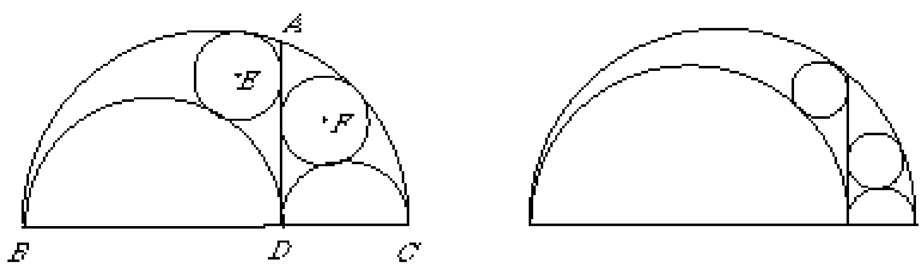

Figure 3.

Another interesting elementary problem about the arbelos is to show (Figure 4) that, if A and C are the points of contact of the common tangent of the two inner circles then $ABCD$ is a rectangle. See Stueve [12], for example.

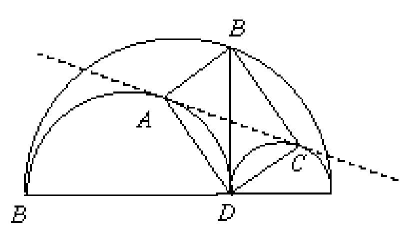

Figure 4.

Five centuries later, Pappus of Alexandria (circa 320 AD) gave a proof of the following remakable theorem about the arbelos. He claimed that it was known to the ancients but we now ascribe to Pappus himself. The rest of this paper is devoted to a proof of this result; the argument is developed from the proof presented by Heath [5] page 371.

Theorem 1 *If a chain of circles $C_1, C_2, \ldots$ is constructed inside an arbelos so that C_1 is tangent to the three semicircles and C_n is tangent to two semi-circles and C_{n-1} then, for all n, the height of the centre of C_n above the base of the arbelos BC is n times the diameter of C_n.*

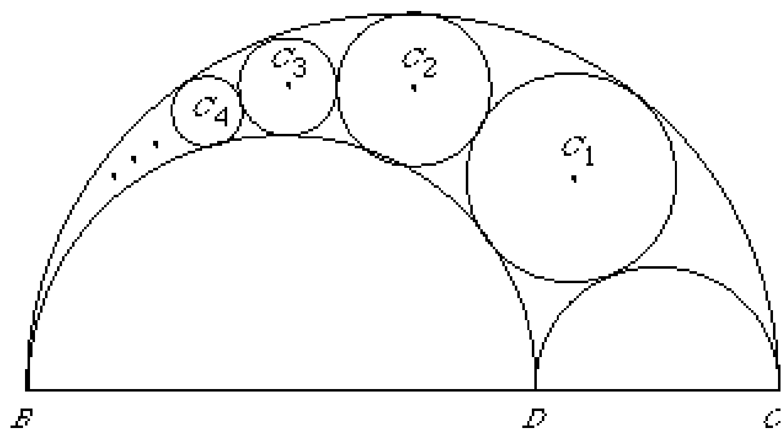

Figure 5. Pappus Theorem

This theorem is even true for $n = 0$ if we allow one of the semicircles to play the role of C_0. The proof of Theorem 1 is by induction on n. The induction step is established in Theorem 2 which is concerned with Figure 6.

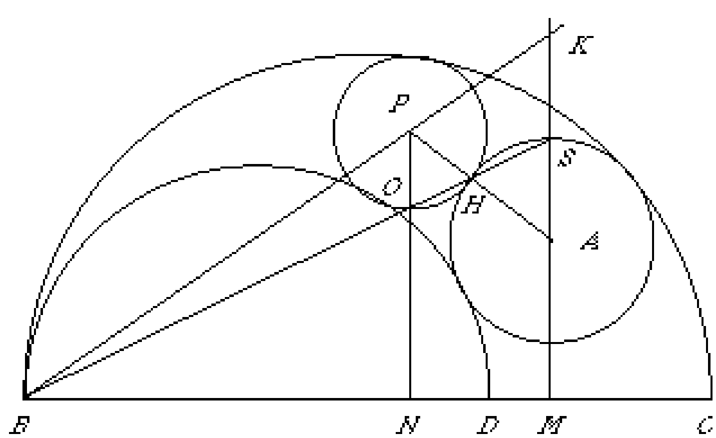

Figure 6.

In the figure, the circles C_P and C_A centred at P and A are tangent to each other, at H, and to the two semicircles. The points N, M are the feet of the perpendiculars from the centres P and A. The line PN

meets C_P in O and the line AM meets C_A in S. The point K is the intersection of BP and MA. The figure shows B, O, H, S as collinear; this requires a proof and is the second claim of the theorem.

Theorem 2 *Let d and d' denote the diameters of C_A and C_P respectively. Then*

(i) $\dfrac{BM}{BN} = \dfrac{d}{d'}$;

(ii) Points B, O, H, S are collinear;

(iii) $\dfrac{AM}{d} + 1 = \dfrac{PN}{d'}$.

If we replace C_A and C_P with consecutive circles C_{n-1} and C_n from Theorem 1 then Part (iii) is exactly what we need for induction. We will prove the results in the order (iii), (i), (ii).

Before embarking on the proof of Theorem 2, we record the following elementary facts about circles which will be needed in the proof.

Lemma 1 *If two circles share a tangent at at their intersection point P then the lines joining the ends of a diameter of one circle to the point P meet the other circle in the ends of a parallel diameter.*

Lemma 2*Suppose that FB is tangent to a circle at B and that FUV is a secant of the same circle. Then $FB^2 = FU \cdot FV$. In particular, the product $FU \cdot FV$ is the same for every secant of the circle through F.*

Proof of Part (iii) using Part (ii) Since $PN||KN$ there are many pairs of similar triangles with a vertex at B. Hence

$$\frac{BM}{BN} = \frac{BK}{BP} = \frac{KS}{PO}.$$

By Part (ii),

$$\frac{KS}{PO} = \frac{d}{d'} = \frac{AS}{PO}$$

since $AS = \frac{1}{2}d$ and $PO = \frac{1}{2}d'$. Thus $KS = AS$. Also

$$\triangle BMK \sim \triangle BNP \quad \Rightarrow \quad \frac{MK}{PN} = \frac{BM}{BN} = \frac{KS}{PO}.$$

But $MK = AM + d$ since $AS = KS = \frac{1}{2}d$. So

$$\frac{MK}{KS} = \frac{PN}{PO} \quad \Rightarrow \quad \frac{AM + d}{\frac{1}{2}d} = \frac{PN}{\frac{1}{2}d'}.$$

Therefore $\dfrac{AM}{d} + 1 = \dfrac{PN}{d'}$. $\square$

Proof of Part (i) To simplify the equations let $r = \frac{1}{2}d$ and $r' = \frac{1}{2}d'$. The idea is to show that

$$\frac{BM}{r} = \frac{BC + BD}{BC - BD}. \qquad (\star)$$

In this equation, the right side does not depend on the circles C_A, C_P at all while the left side depends on only one of these circles. Setting this up for each of the circles in Figure 5 in turn we easily deduce that $\dfrac{BM}{BN} = \dfrac{r}{r'} = \dfrac{d}{d'}$ which is Part (i). The proof of $(\star)$ uses Figure 7.

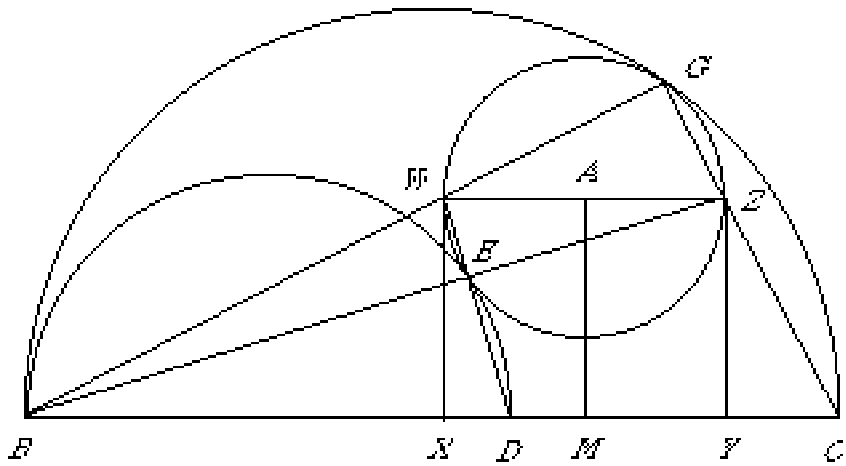

Figure 7.

In Figure 7, we have drawn the diameter WAZ parallel to BC. Then WX, AM, and ZY are all perpendicular to BC.

(a) The figure is correct. That is, the lines WD and BZ intersect at the point E of tangency of C_A and the smaller semi-circle. Similarly the lines BW and CZ meet at the point G of tangency of C_A and the larger semicircle. This follows from Lemma 1.

(b) $\dfrac{BM}{r} = \dfrac{BC+BD}{BC-BD}$. We have

$$\triangle BWX \sim \triangle BCG \Rightarrow \frac{BC}{BG} = \frac{BW}{BX} \Rightarrow BC \cdot BX = BW \cdot BG,$$

and

$$\triangle BYZ \sim \triangle BED \Rightarrow \frac{BZ}{BY} = \frac{BD}{BE} \Rightarrow BD \cdot BY = BE \cdot BZ.$$

But $BW \cdot BG = BE \cdot BZ$ since BWG and BEZ are secants of the same circle. Therefore

$$BC \cdot BX = BD \cdot BY \Rightarrow \frac{BC}{BD} = \frac{BY}{BX}.$$

It is a basic fact about ratios that if $\dfrac{a}{b} = \dfrac{c}{d}$ then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$. Denote by r the radius of C_A. Then $BM = BX + r = BY - r$ so $BX + BY = 2BM$. Hence

$$\frac{BC+BD}{BC-BD} = \frac{BY+BX}{BY-BX} = \frac{2BM}{XY} = \frac{2BM}{2r} = \frac{BM}{r}.$$

This is what we set out to show. $\square$

Proof of Part (ii) using Part (i) The points O, H, S are collinear by Lemma 1. So the whole proof is complete once we show that this line OHS passes through B.

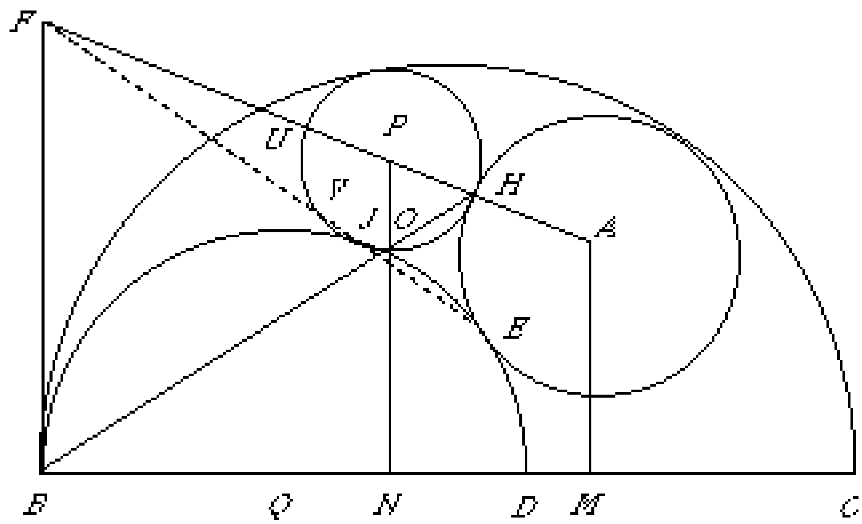

Figure 8.

In Figure 8, F is the intersection of the perpendicular st B and the line through the centres P and A. Also V is the other point of intersection of EJ with C_P and Q is the centre of the circle with diameter BD. We will have to establish that EJ does indeed pass through F. Denote the radii of C_A and C_P by r and r' respectively.

Let O' be the intersection of the lines BH and PN. What we want to prove is that BH contains the point O of intersection of PN and the circle C_P. Thus it is enough to show that$O = O'$ which will follow if $PO' = PH = r'$. Since $FB \| PN$, we have $\triangle HFB \sim \triangle HPO'$. Thus $PO' = PH$ if $FB = FH$.

We can identify F as the point on AP satisfying $\dfrac{FA}{FP} = \dfrac{r}{r'}$. Indeed since $FB \| PN \| AM$, we have $\dfrac{FA}{FP} = \dfrac{BM}{BN}$ and from Theorem 2 Part (i), we have $\dfrac{BM}{BN} = \dfrac{r}{r'}$.

Now let F' be the intersection of EJ with AP, see Figure 9. We want to show that $F = F'$.

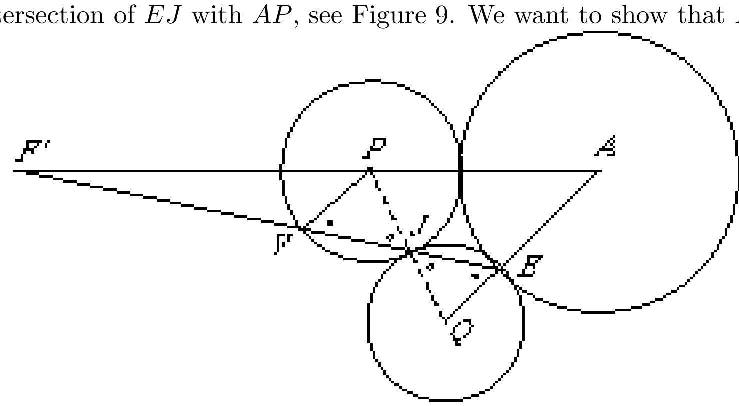

Figure 9.

Since $\triangle PVJ$ and $\triangle QJE$ are each isoceles, it follows that $PV \| AE$. Then $\triangle F'VP \sim \triangle F'EA$. Thus

$$\frac{F'A}{F'P} = \frac{AE}{PV} = \frac{r}{r'}.$$

Since this is the ratio that determines F, we have $F = F'$ and EJ passes through F.

Let $\sigma = \dfrac{r}{r'}$. We have just shown that $\triangle FVP$ and $\triangle FEA$ are similar with ratio σ. It is then straighforward that $\triangle FUV$ and $\triangle FHE$ are similar with the same ratio. Thus $FE = FV \cdot \sigma$ and $FH = FU \cdot \sigma$. Also Lemma 2, applied to two different circles, implies that $FB^2 = FJ \cdot FE$ and $FJ \cdot FV = FH \cdot FU$. Therefore

$$FB^2 = FJ \cdot FE = FJ \cdot FV \cdot \sigma = FH \cdot FU \cdot \sigma = FH^2.$$

Therefore $FB = FH$ so $PO' = PH = PO$ and the point O is indeed on line BH. This completes the proof of Part (ii) (and Theorem 2). $\square$

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e–References

10. Woo, P. <www-students.biola.edu/~woopy/arbelos.htm> Note: This page includes an applet that combines and animates several theorems about the arbelos.

11. Stueve,L. <jwilson.coe.uga.edu/EMT725/class/Stueve/Stueve.html> Note: This page poses an interesting and not too difficult problem about the arbelos.

12. Bogomolny, A. <www.cut-the-knot.com/proofs/arbelos.html> Note: This site <www.cut-the-knot/> is devoted to "Interactive Mathematics Miscellany and Puzzles". It includes the Arbelos under"Attractive Facts"